On the relation between the Maxwell system and the Dirac equation

VLADISLAV V. KRAVCHENKO
Depto. de Telecomunicaciones
SEPI, ESIME Zacatenco
Instituto Politécnico Nacional
Av.IPN S/N, C.P.07738, D.F.
MEXICO
vkravche@maya.esimez.ipn.mx

Abstract

A simple relation between the Maxwell system and the Dirac equation based on their quaternionic reformulation is discussed. We establish a close connection between solutions of both systems as well as a relation between the wave parameters of the electromagnetic field and the energy of the Dirac particle.

Key-Words: - Maxwell equations, Dirac equation

Introduction

The relation between the two most important in mathematical physics first order systems of partial differential equations is among those topics which attract attention because of their general, even philosophical significance but at the same time do not offer much for the solution of particular problems concerning physical models. The Maxwell equations can be represented in a Dirac like form in different ways (e.g., [3], [5], [9]). Solutions of Maxwell's system can be related to solutions of the Dirac equation through some nonlinear equations (e.g., [11]). Nevertheless, in spite of these significant efforts there remain some important conceptual questions. For example, what is the meaning of this close relation between the Maxwell system and the Dirac equation and how this relation is connected with the wave-particle dualism. In the present article we propose a simple equality involving the Dirac operator and the Maxwell operators (in the sense which is explained below). This equality establishes a direct connection between solutions of the two systems and moreover, we show that it is valid when a quite natural relation between the frequency of the electromagnetic wave and the energy of the Dirac particle is fulfilled. Our analysis is based on the quaternionic form of the Dirac equation obtained in [7] and on the quaternionic form of the Maxwell equations proposed in [6] (see also [8]). In both cases the quaternionic reformulations are completely equivalent to the traditional form of the Dirac and Maxwell systems.

Preliminaries

The algebra of complex quaternions is denoted by $\mathbb{H}(\mathbb{C})$. Each complex quaternion a is of the form $a = \sum_{k=0}^{3} a_k i_k$ where $\{a_k\} \subset \mathbb{C}$, i_0 is the unit and $\{i_k | \; k = 1,2,3\}$ are the quaternionic imaginary units:

$$i_0^2 = i_0 = -i_k^2; \; i_0 i_k = i_k i_0 = i_k, \quad k = 1,2,3;$$

$$i_1 i_2 = -i_2 i_1 = i_3; \; i_2 i_3 = -i_3 i_2 = i_1;$$

$$i_3 i_1 = -i_1 i_3 = i_2.$$

The complex imaginary unit i commutes with i_k, $k = \overline{0,3}$.

We will use the vector representation of complex quaternions: $a = \mathrm{Sc}(a) + \mathrm{Vec}(a)$, where $\mathrm{Sc}(a) = a_0$ and $\mathrm{Vec}(a) = \overrightarrow{a} = \sum_{k=1}^{3} a_k i_k$. That is each complex quaternion is a sum of its scalar part and its vector part. Complex vectors we identify with complex quaternions whose scalar part is equal to zero. In vector terms, the multiplication of two arbitrary complex quaternions a and b can be written as follows:

$$a \cdot b = a_0 b_0 - < \overrightarrow{a}, \overrightarrow{b} > + \left[\overrightarrow{a} \times \overrightarrow{b}\right] + a_0 \overrightarrow{b} + b_0 \overrightarrow{a},$$

where

$$< \overrightarrow{a}, \overrightarrow{b} > := \sum_{k=1}^{3} a_k b_k \in \mathbb{C}$$

and

$$[\overrightarrow{a} \times \overrightarrow{b}] := \begin{vmatrix} i_1 & i_2 & i_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \in \mathbb{C}^3.$$

We shall consider continuously differentiable $\mathbb{H}(\mathbb{C})-$valued functions depending on three real variables $x = (x_1, x_2, x_3)$. On this set the well known (see, e.g., [1], [4], [8]) Moisil-Theodoresco operator is defined by the expression

$$D := \sum_{k=1}^{3} i_k \partial_k, \qquad \text{where} \qquad \partial_k = \frac{\partial}{\partial x_k}.$$

The action of the operator D on an $\mathbb{H}(\mathbb{C})-$valued function f can be written in a vector form:

$$Df = -\operatorname{div} \overrightarrow{f} + \operatorname{grad} f_0 + \operatorname{rot} \overrightarrow{f}. \qquad (1)$$

That is, $\operatorname{Sc}(Df) = -\operatorname{div} \overrightarrow{f}$ and $\operatorname{Vec}(Df) = \operatorname{grad} f_0 + \operatorname{rot} \overrightarrow{f}$. In a good number of physical applications (see [4] and [8]) the operators $D_\alpha = D + M^\alpha$ and $D_{-\alpha} = D - M^\alpha$ are needed, where α is a complex quaternion and M^α denotes the operator of multiplication by α from the right-hand side: $M^\alpha f = f \cdot \alpha$. Here we will be interested in two special cases when α is a scalar, that is $\alpha = \alpha_0$ or when α is a vector $\alpha = \overrightarrow{\alpha}$. The first case corresponds to the Maxwell equations and the second to the Dirac equation.

The Dirac equation

Consider the Dirac equation in its covariant form

$$(\hbar(\frac{\gamma_0}{c}\partial_t + \sum_{k=1}^{3} \gamma_k \partial_k) + imc)\Phi(t, x) = 0.$$

For a wave function with a given energy we have $\Phi(t, x) = q(x)e^{i\frac{\mathcal{E}}{\hbar}t}$, where q satisfies the equation

$$(\frac{i\mathcal{E}}{c\hbar}\gamma_0 + \sum_{k=1}^{3} \gamma_k \partial_k + \frac{imc}{\hbar})q(x) = 0. \qquad (2)$$

Denote

$$\mathbb{D} := \frac{i\mathcal{E}}{c\hbar}\gamma_0 + \sum_{k=1}^{3} \gamma_k \partial_k + \frac{imc}{\hbar}.$$

Let us introduce an auxiliary notation $\widetilde{f} := f(t, x_1, x_2, -x_3)$. The transformation which allows us to rewrite the Dirac equation in a quaternionic form we denote as $\mathcal{A}$ and define in the following way [7]. A function $\Phi : \mathbb{R}^3 \to \mathbb{C}^4$ is transformed into a function $F : \mathbb{R}^3 \to \mathbb{H}(\mathbb{C})$ by the rule

$$\begin{aligned} F \;=\; & \mathcal{A}[\Phi] = \frac{1}{2}(-(\widetilde{\Phi}_1 - \widetilde{\Phi}_2)i_0 + i(\widetilde{\Phi}_0 - \widetilde{\Phi}_3)i_1 - \\ & (\widetilde{\Phi}_0 + \widetilde{\Phi}_3)i_2 + i(\widetilde{\Phi}_1 + \widetilde{\Phi}_2)i_3). \end{aligned}$$

The inverse transformation $\mathcal{A}^{-1}$ is defined as follows

$$\Phi = \mathcal{A}^{-1}[F] =$$

$$(-i\widetilde{F}_1 - \widetilde{F}_2, -\widetilde{F}_0 - i\widetilde{F}_3, \widetilde{F}_0 - i\widetilde{F}_3, i\widetilde{F}_1 - \widetilde{F}_2).$$

Let us present the introduced transformations in a more explicit matrix form which relates the components of a $\mathbb{C}^4$-valued function Φ with the components of an $\mathbb{H}(\mathbb{C})$-valued function F:

$$F = \mathcal{A}[\Phi] = \frac{1}{2}\begin{pmatrix} 0 & -1 & 1 & 0 \\ i & 0 & 0 & -i \\ -1 & 0 & 0 & -1 \\ 0 & i & i & 0 \end{pmatrix}\begin{pmatrix} \widetilde{\Phi}_0 \\ \widetilde{\Phi}_1 \\ \widetilde{\Phi}_2 \\ \widetilde{\Phi}_3 \end{pmatrix}$$

and

$$\Phi = \mathcal{A}^{-1}[F] = \begin{pmatrix} 0 & -i & -1 & 0 \\ -1 & 0 & 0 & -i \\ 1 & 0 & 0 & -i \\ 0 & i & -1 & 0 \end{pmatrix}\begin{pmatrix} \widetilde{F}_0 \\ \widetilde{F}_1 \\ \widetilde{F}_2 \\ \widetilde{F}_3 \end{pmatrix}.$$

We have the following important equality

$$D_{\overrightarrow{\alpha}} = -\mathcal{A}\gamma_1\gamma_2\gamma_3\mathbb{D}\mathcal{A}^{-1}, \qquad (3)$$

where $\overrightarrow{\alpha} := -\frac{1}{\hbar}(i\frac{\mathcal{E}}{c}i_1 + mci_2)$. This equality shows us that instead of equation (2) we can consider the equation

$$D_{\overrightarrow{\alpha}} f = 0 \qquad (4)$$

and the relation between solutions of (2) and (4) is established by means of the invertible transformation $\mathcal{A}$: $f = \mathcal{A}q$.

The Maxwell equations

We will consider the time-harmonic Maxwell equations for a sourceless isotropic homogeneous medium

$$\operatorname{rot} \overrightarrow{H} = -i\omega\varepsilon \overrightarrow{E}, \qquad (5)$$

$$\operatorname{rot} \overrightarrow{E} = i\omega\mu \overrightarrow{H}, \qquad (6)$$

$$\operatorname{div} \overrightarrow{E} = 0, \qquad (7)$$

$$\operatorname{div} \overrightarrow{H} = 0. \qquad (8)$$

Here ω is the frequency, ε and μ are the absolute permittivity and permeability respectively. $\varepsilon = \varepsilon_0\varepsilon_r$ and $\mu = \mu_0\mu_r$, where ε_0 and μ_0 are the corresponding parameters of a vacuum and ε_r, μ_r are the relative permittivity and permeability of a medium.

Taking into account (1) we can rewrite this system as follows

$$D\overrightarrow{E} = i\omega\mu\overrightarrow{H}, \tag{9}$$

$$D\overrightarrow{H} = -i\omega\varepsilon\overrightarrow{E}. \tag{10}$$

This pair of equations can be diagonalized in the following way [6] (see also [8]). Denote

$$\overrightarrow{\varphi} := -i\omega\varepsilon\overrightarrow{E} + \kappa\overrightarrow{H} \tag{11}$$

and

$$\overrightarrow{\psi} := i\omega\varepsilon\overrightarrow{E} + \kappa\overrightarrow{H}, \tag{12}$$

where $\kappa := \omega\sqrt{\varepsilon\mu} = \frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}$ is the wave number. Applying the operator D to the functions $\overrightarrow{\varphi}$ and $\overrightarrow{\psi}$ one can see that $\overrightarrow{\varphi}$ satisfies the equation

$$(D-\kappa)\overrightarrow{\varphi} = 0, \tag{13}$$

and $\overrightarrow{\psi}$ satisfies the equation

$$(D+\kappa)\overrightarrow{\psi} = 0. \tag{14}$$

Solutions of (13) and (14) are called the Beltrami fields (see, e.g., [10]).

The relation

In the preceding sections it was shown that the Dirac equation (2) is equivalent to the equation $D_{\overrightarrow{\alpha}}f = 0$ with $\overrightarrow{\alpha} = -\frac{1}{\hbar}(i\frac{\mathcal{E}}{c}i_1 + mci_2)$ and the Maxwell equations (5)-(8) are equivalent to the pair of quaternionic equations $D_{-\kappa}\overrightarrow{\varphi} = 0$ and $D_\kappa\overrightarrow{\psi} = 0$. Now we will show a simple relation between these objects. Suppose that

$$\kappa^2 = \overrightarrow{\alpha}^2. \tag{15}$$

Let us introduce the following operators of multiplication

$$P^\pm := \frac{1}{2\kappa}M^{\kappa\pm\overrightarrow{\alpha}}.$$

It is easy to verify that they are mutually complementary and orthogonal projection operators, and the following equality is valid [8]

$$D_{\overrightarrow{\alpha}} = P^+D_\kappa + P^-D_{-\kappa}. \tag{16}$$

Moreover, as $P^\pm$ commute with $D_{\pm\kappa}$, we obtain that any solution of (4) is uniquely represented as follows

$$f = P^+\psi + P^-\varphi,$$

where φ and ψ are solutions of (13) and (14) respectively but in general can be full quaternions not necessarily purely vectorial. In particular, we have that

$$f = P^+(i\omega\varepsilon\overrightarrow{E} + \kappa\overrightarrow{H}) + P^-(-i\omega\varepsilon\overrightarrow{E} + \kappa\overrightarrow{H}) =$$

$$i\omega\varepsilon(P^+ - P^-)\overrightarrow{E} + \kappa(P^+ + P^-)\overrightarrow{H} =$$

$$\frac{i\omega\varepsilon}{\kappa}\overrightarrow{E}\cdot\overrightarrow{\alpha} + \kappa\overrightarrow{H}$$

is a solution of (4) if $\overrightarrow{E}$ and $\overrightarrow{H}$ are solutions of (5)-(8).

It should be noticed that (16) works in both directions. We have

$$D_\kappa = P^+D_{\overrightarrow{\alpha}} + P^-D_{-\overrightarrow{\alpha}}$$

and

$$D_{-\kappa} = P^-D_{\overrightarrow{\alpha}} + P^+D_{-\overrightarrow{\alpha}}.$$

The fact that the Maxwell system reduces to equations (13) and (14), where the functions $\overrightarrow{\varphi}$ and $\overrightarrow{\psi}$ are purely vectorial provokes the natural question whether it had any sense to consider full quaternions φ and ψ and hence four-component vectors E and H or the nature definitely eliminated their scalar parts. Some arguments supporting the idea of nonzero scalar parts can be found, for example, in [2].

As we have seen equality (16) is valid under the condition (15). Let us analyze this condition. Note that

$$\overrightarrow{\alpha}^2 = -<\overrightarrow{\alpha},\overrightarrow{\alpha}> = \frac{1}{\hbar^2}(\frac{\mathcal{E}^2}{c^2} - m^2c^2).$$

Thus (15) has the form

$$\kappa^2 = \frac{1}{\hbar^2}(\frac{\mathcal{E}^2}{c^2} - m^2c^2) \tag{17}$$

or equivalently

$$(\hbar\omega)^2\varepsilon_r\mu_r = \mathcal{E}^2 - m^2c^4.$$

From this equation in the case $\varepsilon_r = \mu_r = 1$, that is for a vacuum, using the well known in quantum mechanics relation between the frequency and the impulse: $\hbar\omega = pc$ we obtain the equality

$$\mathcal{E}^2 = p^2c^2 + m^2c^4. \tag{18}$$

In general, if in (17) we formally use the de Broglie equality $p = \hbar\kappa$, we again obtain the fundamental relation (18).

Thus relation (16) between the Dirac operator and the Maxwell operators is valid if the condition (17) is fulfilled which quite surprisingly is in agreement with (18).

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