A Solution to the Optimal Tracking Problem for Linear Systems

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Abstract: The paper establishes a new procedure to obtain the solution for the optimal tracking problem based on dynamic programming. The optimal control refers to a quadratic criterion with finite final time, regarding a perturbed time-variant linear system. The proposed algorithm can be easier implemented by comparison with other procedures.

Key-Words: optimal control, linear quadratic, tracking problem

1 Introduction

A perturbed linear time-variant multivariable system is considered

$$\begin{aligned} &\dot{x}(t) = A(t)x(t) + B(t)u(t) + w(t), \quad x(t_0) = x^0 \\ &y(t) = C(t)x(t) \end{aligned} \tag{1}$$

where $x(t) \in \Re^n, u(t) \in \Re^m, y(t) \in \Re^r, w(t) \in \Re^n$ are the sate, control, output and disturbance vector, respectively.

The problem is to ensure that the output vector y(t) evolves near to a desired trajectory $z(t) \in \Re^r$ and that the energy consumption has a low level. For this purpose, it is introduced the criterion

$$\begin{aligned} J = &\frac{1}{2} e^T(t_f)Se(t_f) + \\ &+ \frac{1}{2}\int_{t_0}^{t_f} \left[e^T(t)Q(t)e(t) + u^T(t)P(t)u(t)\right]dt \end{aligned} \tag{2}$$

(T denotes the transposition), where S≥0, Q(t) ≥0, P(t)>0 are the weight matrices of appropriate dimensions and

$$e(t) = z(t) - y(t) = z(t) - C(t)x(t) \tag{3}$$

is the tracking error.

The optimal tracking problem refers to the system (1) and the criterion (2). If the pair (A,C) is completely observable, the problem can be reformulated as one referring to the state vector [1], and thus the criterion is

$$\begin{aligned} J = &\frac{1}{2} x^T(t_f)S'x(t_f) + \frac{1}{2} z^T(t_f)Sz(t_f) - \\ &- z^T(t_f)SCx(t_f) + \\ &+ \frac{1}{2}\int_{t_0}^{t_f} \left[x^T(t)Q'(t)x(t) + z^T(t)Q(t)z(t) - \right. \\ &\left. - 2z^T(t)Q(t)C(t)x(t) + u^T(t)P(t)u(t)\right]dt \end{aligned} \tag{4}$$

where

$$\begin{aligned} &S' = C^T(t_f)QC(t_f) \geq 0, \\ &Q'(t) = C^T(t)Q(t)C(t) \geq 0 \end{aligned} \tag{5}$$

Let us denote with V(t,x) the minimum value of the criterion (4) on the interval [t, t_f]. This function satisfies the Hamilton-Jacobi-Bellman equation [2]

$$\frac{\partial V(t,x)}{\partial t} + H\left(\frac{\partial V(t,x)}{\partial x}, x, t\right) = 0 \tag{6}$$

and

$$\begin{aligned} V(t_f, x) = &\frac{1}{2} x^T(t_f)S'x(t_f) + \frac{1}{2} z^T(t_f)Sz(t_f) - \\ &- z^T(t_f)SCx(t_f) \end{aligned} \tag{7}$$

For the above formulated problem

$$\begin{aligned} H(p,x,t,u) = &\frac{1}{2}\left(x^T Q' x + u^T P u\right) + \\ &+ p^T(Ax + Bu + w), \quad p(t) \in \Re^n \end{aligned} \tag{8}$$

The optimal control vector is

$$u(t) = -P^{-1}(t)B^T(t)p(t) \tag{9}$$

If V(t,x) is imposed as

$$V(t,x) = \frac{1}{2}x^T(t)\tilde{R}(t)x(t) + x^T(t)\tilde{g}(t) \tag{10}$$

($\tilde{g}(t)$ depends on z(t)), thus V(t,x) is the solution of equation (3) only if the symmetric matrix $\tilde{R}$ verifies the Riccati differential equation

$$\begin{aligned}\dot{\tilde{R}}(t) = \tilde{R}(t)B(t)P^{-1}(t)B^T(t)\tilde{R}(t) - \tilde{R}(t)A(t) - \\ - A^T(t)\tilde{R}(t) - Q(t)\end{aligned} \tag{11}$$

with

$$\tilde{R}(t_f) = S' \tag{12}$$

and $\tilde{g}(t)$ satisfies a linear differential equation.

Finally, the optimal control becomes

$$u(t) = -P^{-1}(t)B^T(t)[\tilde{R}(t)x(t) + \tilde{g}(t)] \tag{13}$$

Matrix $\tilde{R}(t)$ is time variant even in the cases when the matrices of the system (1) and of the criterion (2) are constant. It means that the resulting optimal controller is time variant even in the case of an invariant linear quadratic tracking problem. Solving in inverse time of the equation (11) introduces a supplementary difficult in the implementation of this controller.

In order to avoid these difficulties, a new procedure is proposed. This procedure is similar with one proposed in [3] for a state regulator problem for an unperturbed system. Note that the presence of exogenous vectors z(t) and w(t) complicates the solution, since the additional terms appear. The proposed procedure is especially useful in the time invariant case, when a significant simplification of the implementation is obtained.

2 Main results

Instead of (10), it is proposed V(t,x) in the form

$$V(t,x) = \frac{1}{2}x^T(t)\overline{R}(t)x(t) + x^T v(t) + \eta(t) \tag{14}$$

where the symmetric matrix $\overline{R}(t)$ is a particular solution to the equation (11), and we denote

$$\overline{R}(t_f) = \overline{S} \tag{15}$$

Note that, in many cases, such a solution can be beforehand computed off-line and then used in the real time computing.

Then

$$\begin{aligned}\frac{\partial V(t,x)}{\partial t} &= \frac{1}{2}x^T\dot{\overline{R}}(t)x + x^T\dot{v}(t) + \dot{\eta}(t) \\ \frac{\partial V(t,x)}{\partial x} &= \overline{R}x + v(t)\end{aligned} \tag{16}$$

and, using (8), (9) and (16), yields

$$\begin{aligned}&H(\frac{\partial V(t,x)}{\partial t}, x, t) = \frac{1}{2}x^T Q' x - \\ &- \frac{1}{2}(\overline{R}x + v)^T N(\overline{R}x + v) + \frac{1}{2}(\overline{R}x + v)^T Ax + \\ &+ \frac{1}{2}x^T A^T(\overline{R}x + v) + \frac{1}{2}z^T Qz - z^T QCx + \\ &+ (\overline{R}x + v)^T w \quad ,\end{aligned} \tag{17}$$

$$\text{where } N(t) = B(t)P^{-1}(t)B^T(t) \tag{18}$$

Lemma 1: The minimum cost function V(t,x) can be written in the form (14), where $\overline{R}(t)$ is a particular solution to the Riccati equation (11), with final condition (15). The vector v(t) and the scalar $\eta(t)$ satisfy the equations:

$$\dot{v}(t) = -F^T v(t) + h(t) \tag{19}$$

$$v(t_f) = (S' - \overline{S})x(t_f) - SCz(t_f) \tag{20}$$

where

$$F(t) = A(t) - N(t)R(t) \tag{21}$$

$$h(t) = C^T(t)Q(t)z(t) - \overline{R}(t)w(t) \tag{22}$$

and

$$\begin{aligned}\dot{\eta}(t) = \frac{1}{2}v^T(t)N(t)v(t) - v^T(t)w(t) - \\ - \frac{1}{2}z^T(t)Q(t)z(t)\end{aligned} \tag{23}$$

$$\begin{aligned}\eta(t_f) = -\frac{1}{2}x^T(t_f)(S' - \overline{S})x(t_f) + \\ + \frac{1}{2}z^T(t_f)Sz(t_f)\end{aligned} \tag{24}$$

Proof: Substituting (16) and (17) in (6), yields

$$\frac{1}{2}x^T(\dot{\overline{R}} - \overline{R}N\overline{R} + \overline{R}A + A^T\overline{R} + Q')x +$$
$$+ x^T[\dot{v} + (A^T - \overline{R}N)v - C^TQz + \overline{R}w] + \dot{\eta} +$$
$$+ \frac{1}{2}z^TQz + v^Tw - \frac{1}{2}v^TNv = 0$$

This relation is satisfied for any x, if $\overline{R}(t)$ is solution to the equation (11) and v(t) and $\eta(t)$ satisfy (19) and (23), respectively. The final condition (7) is also satisfied. ■

Let us denote with $\Psi(t, t_f)$ and $\Phi(t, t_f)$ the transition matrices for F and $-F^T$, respectively. The solution to the equation (19) is

$$v(t) = \Phi(t, t_f)v(t_f) + \int_{t_f}^{t} \Phi(\tau, t_f)h(\tau)d\tau \tag{25}$$

This solution cannot be used because the final condition $v(t_f)$ is not a priori known. Therefore the final condition $v(t_f)$ must be expressed as a function of $x(t_0)$. In this respect, we observe that the co-state vector

$$p(t) = \overline{R}(t)x(t) + v(t) \tag{26}$$

can be replaced in (9) and then in (1). Thus the equations (1) and (24) can be written:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = G\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} w(t) \\ h(t) \end{bmatrix}, \tag{27}$$

with

$$G(t) = \begin{bmatrix} F(t) & -N(t) \\ 0 & -F^T(t) \end{bmatrix} \in \Re^{2nx2n} \tag{28}$$

where N and F are given by (18) and (21), respectively.

The transition matrix for G(t) can be expressed as [3]:

$$\Omega(t, t_f) = \begin{bmatrix} \Psi(t, t_f) & \Omega_{12}(t, t_f) \\ 0 & \Phi(t, t_f) \end{bmatrix} \tag{29}$$

where $\Psi(t)$ and $\Phi(t)$ are the transition matrices for F and $-F^T$, respectively, and Ω_{12} satisfies the equation

$$\begin{aligned} &\dot{\Omega}_{12}(t, t_f) = F(t)\Omega_{12}(t, t_f) - N\Phi(t, t_f), \\ &\Omega_{12}(t_f, t_f) = 0 \end{aligned} \tag{30}$$

and it is

$$\Omega_{12}(t, t_f) = \int_{t}^{t_f} \Psi(t, \tau)N(\tau)\Phi(\tau, t_f)d\tau \tag{31}$$

The solution to the system (27) is

$$\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \Omega(t, t_f)\begin{bmatrix} x(t_f) \\ v(t_f) \end{bmatrix} + \int_{t_f}^{t} \Phi(t, \tau)\begin{bmatrix} w(\tau) \\ h(\tau) \end{bmatrix}d\tau \tag{32}$$

with $\Omega(t, t_f)$ given by (29).

We are now in position to formulate the following

<u>Theorem</u>: for the formulated tracking problem, the optimal control is

$$u^*(t) = u_f(t) + u_c(t), \tag{33}$$

where $u_f(t)$is the feedback component

$$u_f(t) = -P^{-1}B^T(t)\overline{R}(t)x(t) \tag{34}$$

and $u_c(t)$ is a corrective component

$$u_c(t) = -P^{-1}B^Tv(t) \tag{35}$$

The vector v(t) has two components

$$v(t) = v_0(t) + v_e(t), \tag{36}$$

one depending on the initial state $x(t_0)$, and one depending on the exogenous variables

$$\begin{aligned} &v_0(t) = \Phi(t, t_0)W(t_0, t_f)(S' - \overline{S})M^{-1}(t_0, t_f)x(t_0) \\ &v_e(t) = \Phi(t, t_0)W(t_0, t_f)[\Omega_{12}(t_0, t_f)SCz(t_f) \\ &-g_2(t_0, t_f)] - \Phi(t, t_0)SCz(t_f) + g_1(t, t_f) \end{aligned} \tag{37}$$

where

$$W(t, t_f) = \Phi(t, t_f)(S' - \overline{S})M^{-1}(t, t_f) \tag{38}$$

and $$g_1(t,t_f) = \int_{t_f}^{t} \Phi(t,\tau)h(\tau)d\tau \quad (39)$$

$$g_2(t,t_f) = \int_{t_f}^{t} [\Psi(t,\tau)w(\tau) + \Omega_{12}(t,\tau)h(\tau)]d\tau \quad (40)$$

<u>*Proof*</u>: One can express x(t) from (32) and (20)

$$x(t) = M(t,t_f)x(t_f) - \Omega_{12}(t,t_f)SCz(t_f) + \\ + g_2(t,t_f) \quad (41)$$

where

$$M(t,t_f) = \Psi(t,t_f) + \Omega_{12}(t,t_f)(S' - \overline{S}) \quad (42)$$

and corresponds to the transition from $x(t_f)$ to x(t) for the controlled system without exogenous variables.

From (41) results

$$x(t_f) = M^{-1}(t_0,t_f)[x(t_0) + \\ + \Omega_{12}(t_0,t_f)SCz(t_f) - g_2(t_0,t_f)] \quad (43)$$

Using (43), the expression (25) of v(t) can be written as

$$v(t) = \Phi(t,t_f)\{(S' - \overline{S})M^{-1}[x(t_0) + \\ + \Omega_{12}(t_0,t_f)SCz(t_f) - g_2(t_0,t_f)] - \\ - SCz(t_f)\} + g_1(t,t_f) \quad (44)$$

From (44) it is possible to separate the components (37) depending on the initial state and on the exogenous variables, respectively. ■

<u>Remark 1</u>: Using the above relations, it is proved in [3] that the solution to the Riccati matriceal differential equation (11) is

$$\widetilde{R}(t) = \overline{R}(t) + \Phi(t,t_f)(S' - \overline{S})M^{-1}(t,t_f) \quad (45)$$

<u>Remark 2</u>: The function $g_2(t_0,t_f)$ given by (42) for $t = t_0$ can be computed only if the exogenous vectors z(t) and w(t) are beforehand known. Therefore, the problem can be solved only under this assumption, or, at least, the shape of these vectors is known and their amplitude is estimated at the beginning of the optimisation process.

3 Optimal controller

The classical solution for the optimal tracking problem is based on the relation (13), but the drawbacks specified in the section 1 appear in this case. A simpler controller implementation can be done using the relations presented in this paper.

- A first possibility is to use the implementation based on relation (13), but using also (45) in order to compute the matrix $\widetilde{R}(t)$. Note that particular solution $\overline{R}(t)$ to the equation (13) can be computed beforehand.

- Another way based on (45) is to write this relation so that $\widetilde{R}(t)$ can be computed in direct time. If $\widetilde{R}(t_0)$ would be known, we could write [3]

$$\widetilde{R}(t) = \overline{R} + \Phi(t,t_0)[\widetilde{R}(t_0) - \overline{R}(t_0)]M_0^{-1}(t,t_0)$$

with

$$M_0(t,t_0) = \Psi(t,t_0) + \Omega_{12}(t,t_{0f})[\widetilde{R}(t_0) - \overline{R}(t_0)]$$

But $\widetilde{R}(t_0)$ can be computed taking $t = t_0$ in (45):

$$\widetilde{R}(t_0) = \overline{R}(t_0) + \Phi(t,t_0)(S' - S)M^{-1}(t_0,t_f)$$

Replacing in the previous relation, yields

$$\widetilde{R}(t) = \overline{R}(t) + \Phi(t,t_0)W(t_0,t_f)M_0^{-1}(t_0,t_f)$$

where $W(t_0,t_f)$ is a constant matrix given by (38) for $t = t_0$.

The matrix $\Phi(t,t_0)$ and the matrices from $M_0(t,t_0)$ can be iteratively computed.

-A more convenient way is based on relations offered by the above theorem. These relations are rather complicated, but the most part of the computing is performed off-line in the design stage of the controller. The real-time control implies to compute $u_f(t)$ given by (34) as a usual feedback component and the corrective component $u_c(t)$, given by (35). This last component depends on v(t) given by (44) and contains only two time variables elements : $\Phi(t,t_0)$ and $g_1(t,t_f)$. The both elements can be recurrently computed. Indeed, $\Phi_i = \Phi(t_i,t_0) = \Phi(t_i,t_{i-1})\Phi(t_{i-1},t_0) = \Phi_{\delta i}\Phi_{i-1}$, with $\Phi_0 = \Phi(t_0,t_0) = I$ (the identity matrix). In the above relation Φ_i represents the transition matrix at the moment t_i and $\Phi_{\delta i}$ is the transition matrix which corresponds to two successive moments t_{i-1} and t_i. For the time invariant problem this last matrix is constant.

The vector $g_1(t,t_f)$ given by (39) must be firstly expressed in terms of t_0, in order to ensure a real time computing. In this respect we observe that

$$g_1(t,t_f) = -\int_t^{t_f} \Phi(t,\tau)h(\tau)d\tau = \int_{t_0}^{t} \Phi(t,\tau)h(\tau)d\tau - \int_{t_0}^{t_f} \Phi(t,\tau)h(\tau)d\tau \quad (46)$$

The last integral in (46) is constant and can be computed off-line and the first one can be recurrently computed in real time.

Based on these remarks, results that the real time computing performed by the optimal controller is not very complicated.

A significant simplification of the computing appear in time invariant problems, and moreover in the case when the exogenous vectors are constants.

The solution to the Riccati differential equation in this case is

$$\tilde{R}(t) = R + \Phi(t,t_f)(S' - R)M^{-1}(t,t_f) \quad (47)$$

where R represents a solution to the Riccati algebraic matriceal equation

$$RNR - RA - A^T R - Q = 0 \quad (48)$$

For P>0, Q≥0, (A,B) stabilizable and (A, $\sqrt{Q}$) detectable, the unique positive defined solution of this equation will be chosen.

We also remark that the constant vectors w and h can pull out from the integrals (39) and (42) and this fact simplifies the computing.

We note that in this case the structure of the optimal controller contains only time invariant blocks.

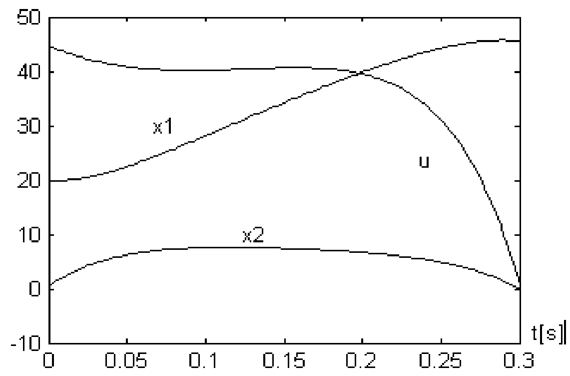
4 Numerical example

As a simple numerical example was chosen a time invariant system described by the equation (1) with

$$A = \begin{bmatrix} 0 & 20 \\ -3.5 & -19.4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 6.3 \end{bmatrix}$$

In the criterion (4), the values of the terminal moments are $t_0 = 0$ and $t_f = 0.3$ and the matrices are chosen as follows:S=diag(10,0), Q=diag(1,3.1), P=p=1. The exogenous vectors are z(t)=[54 2.3] and w(t)=[-28 0].

The optimal controller was designed based on the relations offered by the above theorem. The behaviour of the system is indicated in the figure



5. Conclusions

The optimal tracking problem for a linear time-variant system is studied, tacking into account the presence of the disturbances.

The proposed algorithms are more convenient for implementation by comparing with the usual procedures.

One of the proposed algorithms is based on the direct time solving of the Riccati equation.

It is also indicated an efficient possibility of implementation for the optimal controller, using a usual feedback and a corrective component, depending on the initial state. This optimal controller is advantageous especially in the time-invariant case.

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