Solution of classical non-linear systems using electronic macromodels.

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Abstract: - A brief review of some classical non linear systems is presented. Most of these problems are stated as a set of non linear first order equations or its state space. These systems are simulated using circuital macro models suited for program simulators like Spice.

Key- Words: -Non linear systems, Spice macro models, state space model.

1 Introduction

Simulation of electronic circuits has replaced, many times, breadboarding as a mean of verifying and analyzing the performance of complex circuits or systems. These simulated results depend on how good are the inner models of the simulator programs. Actually the number of elements has far outstripped the capability of Spice [1,2] like simulators, just think in a VLSI chip with thousands or even millions of transistors. Significant research has been done to extend the capabilities of actual program simulators, i.e. nonlinear frequency analysis programs and its steady state solutions, like Harmonica[3].

One solution is to rise the modeling level to a behavioral or macromodel level of modeling. The ability to represent entire circuit blocks by an equation or a set of equations will make simulation of complete and complex analog systems possible, this will speed up the simulation and evaluation of different blocks of analog systems. Modeling entire behavioral or structural blocks like digital design in VHDL, hardware language descriptors. AHDL its analog counterpart (1990, IEEE) is now being developed [4].

In this article a brief review of some classical non linear systems is presented. These problems are stated as a set of non linear first order equations or state equations form and are macro modeled in a program simulator like Spice. The solution and theory of these state Space methods have been well studied, it gives a very good visualization of the physical problem, it can be modeled in a digital or analog computer and gives a general method applicable to discrete and continuous systems, linear and non linear and time variant or not. A brief discussion and some mathematical ideas precede the macromodel solution, this solution is simple in most cases and can be analyzed with different parameters, coefficients or initial conditions, it must be mentioned that there are excellent references dealing with each specific problem, this is just another way to solve and understand these kind of problems, we do not discuss some mathematical points like stability, behavior around some critical points, etc, we make emphasis in a graphical solution, remembering that an image sometimes gives better comprehension than analytical methods.

2 Problem formulation and solution

2.1 Dynamics of a population.

A well known mathematical model that describes the dynamics of a population is the Lotka Volterra model [5].

$$\frac{dN_1}{N\ dt} = f_1(N_1, N_2) \qquad (1)$$

$$\frac{dN_2}{N_2 dt} = f_2(N_1, N_2) \qquad (2)$$

where N_1 is the population density of species 1, N_2 is the density of species2, f_1 and f_2 functions represent an specific natural growing, and depends basically on the birth rate and the mortality index, these equations can be applied to a predator and prey population. In absence of predators, N_1 the prey density will be:

$$\frac{dN_1}{N\ dt} = b - d = \alpha \qquad (3)$$

where *b* is the birth rate and *d* is the mortality rate, α will be positive when there is enough food. With predators present, the prey will be eaten proportional to the number of predators

$$\frac{dN_1}{N\ dt} = \alpha - \beta N_2 \qquad (4)$$

| In absence of preys,N_2 the predator density will die with a rate γ

$$\frac{dN_2}{N_2 dt} = -\gamma \qquad (5)$$

but if there are preys they will survive proportional to the number of preys.

$$\frac{dN_2}{N_2 dt} = -\gamma + \partial N_1 \qquad (6)$$

Depending on the values of α, β, γ, δ and the initial or final values, a different solutions arises, stable oscillations, damped solutions and highly unstable solutions.

An specific example taken from reference [6]:

$$\frac{dx}{dt} = 2x - xy \qquad (7)$$

$$\frac{dy}{dt} = -8y - 2xy \qquad (8)$$

where x describes a small fish population, perhaps sole and y the predator population ie. sharks. This problem can be easily modeled using ideal circuit macromodels, suited for circuit simulators like Spice. In a capacitor:

$$\frac{dv(t)}{dt} = \frac{1}{C} i(t) \qquad (9)$$

i(t) can be expressed as a current source controlled by voltages, VCCS, in a polynomial form. Equations (7,8) can be modeled with two of this models as can be seen in Fig 1, where x and y correspond to the voltages of each circuit.

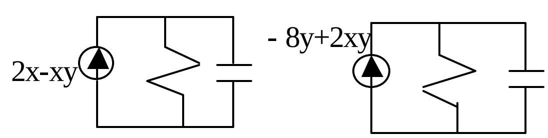

Fig.1 Circuital macromodels of Eqs. 7 and 8.

The Spice netlist is also presented in table 1.

```
VOLTERRA EQUATIONS.
R1 1 0 1e9
R2 2 0 1e9
C1 1 0 1
C2 2 0 1
G1 0 1 POLY(2) 1 0 2 0
+0 2 0 0 –1
G2 0 2 POLY(2) 2 0 1 0
+0 –8 0 0 2
.IC V(1)=1 V(2)=1
.TRAN .1 10 UIC
.Probe
.end
```

Table 1 Spice Netlist.

Spice simulated results of the prey and predator populations are shown in Fig.2 and Fig. 3. The Spice simulated phase plane portrait of the system is shown in Fig.3.

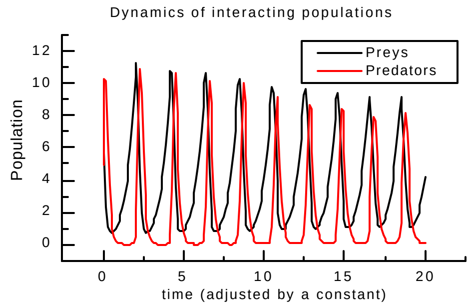

Fig.2 Dynamics of interacting populations.

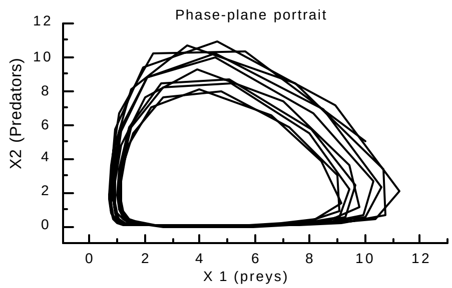

Fig. 3 Phase plane portrait.

2.2 Kinetics of a fermentation process

Other applications, based on the same theory, with different emphasis can be found in the synthesis of some chemical reactions, ie. the fermentation of penicyllin, the grow rate of pseudomonas ovalis and in the synthesis of gluconic acyd [7]. The fermentation of the penicillin can be modeled as :

$$\frac{dy_1}{dt} = b_1 y_1 - \frac{b_1}{b_2} y_1^2 \qquad (10)$$

$$\frac{dy_2}{dt} = b_3 y_1 \qquad (11)$$

Where y_1 is the concentration of cell mass and y_2 represents the synthesis of the penicillin with initial conditions $y_1(0)=0.03$ and $y_2(0)=0$. The process depends on the temperature, affecting the coefficients by

$$b_1 = w_1 \left[\frac{1 - w_2 (q - w_3)^2}{1 - w_2 (25 - w_3)^2} \right] \qquad (12)$$

$$b_2 = w_4 \left[\frac{1 - w_2 (q - w_3)^2}{1 - w_2 (25 - w_3)^2} \right] \qquad (13)$$

$$b_3 = w_5 \left[\frac{1 - w_2 (q - w_6)^2}{1 - w_2 (25 - w_6)^2} \right] \qquad (14)$$

$b_i \geq 0$

From experimental data W_1=13.5 (b_1 at 25° C), W_2=0.005, W_3=30°C, W_4=.94 (b_2 at 25° C), W_5=1.71 (b_3 at 25° C), W_6=20° C, θ=T° C. To maximize the concentration of penicillin at a final time t=1, the Pontryagin maximum principle is applied, it obtains the optimum temperature profile [8].

Spice simulated results of the penicyllin fermentation and its phase portrait are shown in Figs. 4 and 5.

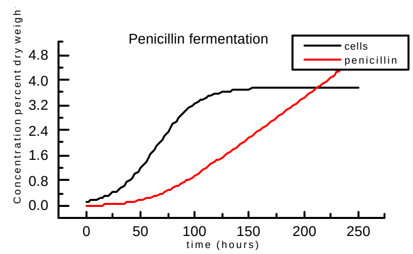

Fig.4 Penicyllin fermentation

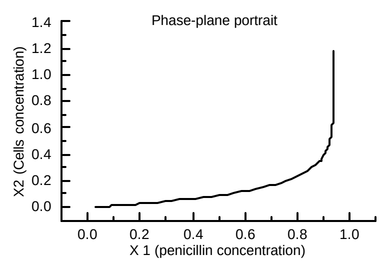

Fig.5 Simulated phase plane portrait.

2.3 Dynamics of volume of flow

. A classical non linear hydraulic example is presented when the flow between two cascaded tanks of water is derived, Fig.6.

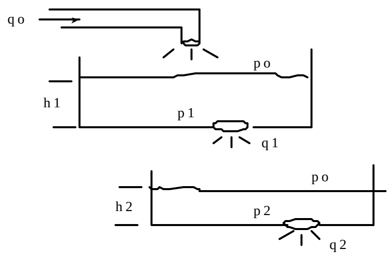

Fig. 6 Water flow between two tanks.

Applying the Bernoulli principle to one tank [9]. The flow of water Q(t) can be written as

$$Q_2(t) = \boldsymbol{b}\sqrt{p_2(t) - p_0} \qquad (15)$$

$$Q_1(t) = \boldsymbol{b}\sqrt{p_1(t) - p_0} \qquad (16)$$

Where β is a constant and depends on the geometry of the outlet hole. P_1 and P_2 , are the internal pressures of tank 1 and tank 2 respectively. The relationship between the pressures and the height of the liquid are:

$$p_1(t) = \boldsymbol{r}gh_1(t) + p_0 \qquad (17)$$

$$p_2(t) = \boldsymbol{r}gh_2(t) + p_0 \qquad (18)$$

Where ρ is the liquid density and g is the gravity constant. In each tank the quantity of water that enters it minus the out flowing one should be the accumulated one, in mathematical terms.

$$\int_{-\infty}^{t} Q_0(t)dt - \int_{-\infty}^{t} Q_1(t)dt = Ah_1 \qquad (19)$$

$$\int_{-\infty}^{t} Q_1(t)dt - \int_{-\infty}^{t} Q_2(t)dt = Ah_2 \qquad (20)$$

Where A is the horizontal section of each tank. The dynamic of the system can be described by:

$$\frac{dh_1(t)}{dt} = \frac{1}{A}\left[Q_0(t) - b\sqrt{rg}\sqrt{h_1(t)}\right] \qquad (21)$$

$$\frac{dh_2(t)}{dt} = \frac{1}{A}\left(\boldsymbol{b}\sqrt{\boldsymbol{r}g}\right)\left[\sqrt{h_1(t)} - \sqrt{h_2(t)}\right] \qquad (22)$$

To simulate this example we make all constants equal to 1 and define a circuital macromodel that extracts the square root of the input, shown in table 2.

```
.subckt csqrt 1 2
*y=x+y-y^2
R1 1 0  1e9
Eout 2 0 poly(2) 1 0 2 0
+0 1  1 0 0 –1
Rout 2 0 1e9
.ends
```

Table 2 Spice netlist of square root extractor.

The simulated results are shown in Figs.7 and 8.

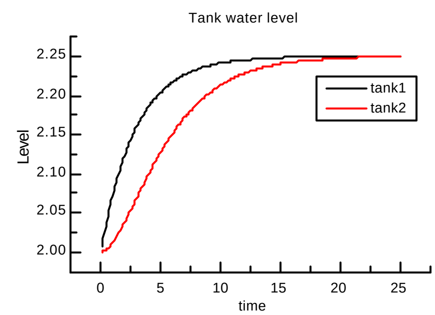

Fig. 7 Tank water level.

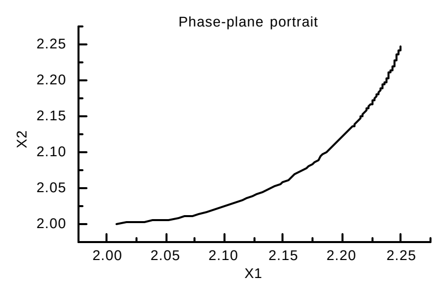

Fig. 8 Phase plane portrait.

2.4. A non linear feedback control system.

In the next system, a classical example of nonlinear feedback control [10], nonlinearities are introduced by the sign function. The behavior of the system can be observed in the state-space plot, Fig.10, showing dx_2/dx. Altough there is no external input to the system, nevertheless it does not reach a steady state, but instead shows a periodic behavior appearing as a limit cycle in the state-space plot.

$$\frac{dx_1}{dt} = -\text{sgn}(x_2) \quad (23)$$

$$\frac{dx_2}{dt} = \text{sgn}(x_1) \quad (24)$$

To obtain the sign function , the subcircuit, shown in table 3, was defined. Simulated results of transient output and phase plane portrait are shown in Figs. 9 and 10 respectively.

```
.subckt sgnx 100 107
Rin 100 0 1e9
E1 102 0 poly (3) 100 0 107 0 104 0
+0 100 –100 1e-15
R1 102 0 1
E2 103 0 102 0 1e6
R2 103 104 1 e12
D1 104 105 diode
V1 105 0 1
D2 106 104 diode
V2 106 0 –1
.model diode D(N=.001)
E4 107 0 104 0 1
.ends
```

Table 3 Spice netlist of sign function.

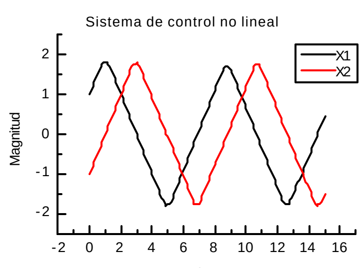

Fig.9 Time response of the system.

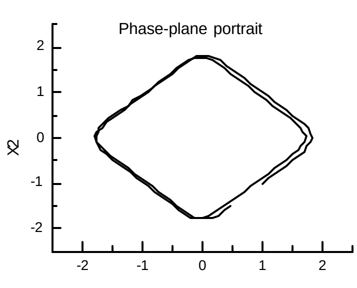

Fig. 10 Phase plane portrait

2.5 A Chaos example, Lorentz Equation

Another classical example where the non linearity gives us a chaotical behaviour is presented with the classical Lorentz Equation [11]. A brief explanations follows.

Human kind has always depend on a variable unstable dynamic atmosphere. From the early beginnings man try to predict weather with all its available tools. Von Neumann in the forties studied not only to predict it, but also to control it. The atmosphere acts like a shield against solar radiation and cosmic energy. It interchanges heat and cold at a huge scale, from equator to poles and from sea level to high altitude. It obeys strictly deterministic laws but ought to the great number of complex factors, it represents a classical chaotic system, non linear feedback generates unpredictable conditions. Weather conditions depend on the solar energy that incidences the planet surface, almost 43% of this energy is in the form of heat energy, the rest remains at the atmosphere or is reflected to the outer space. The heat gradient in the atmosphere moves air masses at a global scale, warmer air with less density ascends faster than cold air, creating regions of high and low pressure. The planet is not circular, rotates around its axe (Coriolis force) and presents an inclination, so solar radiation varies along the year. Lorentz developed a weather model in the 60's, based on differential equations.

A climatic condition could be represented as a point in a 3D space, i.e. x represents the temperature , y humidity and z the barometric pressure. Just plotting the observations along time patterns will show future behavior. He takes Navier Stokes equations in fluid theory. The following results shows the so called Lorentz attractor, an attractor that resembles a butterfly.

$$\frac{dx}{dt} = -10\,x + 10\,y \quad (25)$$

$$\frac{dy}{dt} = 28\,x - y - xz \quad (26)$$

$$\frac{dz}{dt} = xy - \frac{8z}{3} \quad (27)$$

This system shows great sensibility to the values of initial conditions, it tries to describe a phenomenon where little changes in recursive system magnifies and alters drastically the expected results. The well known Lorentz attractors are obtained in Figs. 11 and 12 by drawing x vs. y, and x vs. z respectively.

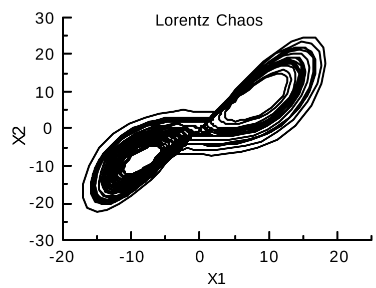

Fig.11 Lorentz attractor.

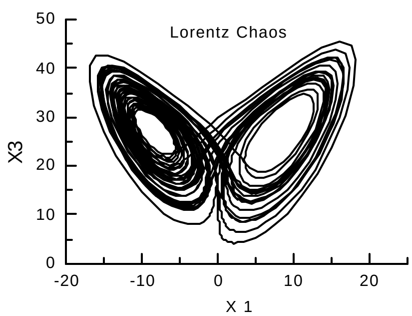

Fig. 12 Lorentz butterfly attractor.

2.6 A Median filter.

In many signal processing applications the suppression of unwanted components can be achieved by linear filters, but in signals with sharp edges corrupted by noise, linear filters also smoothes out signal edges (information) and in addition, impulsive noise cannot be sufficiently suppressed. Another type of filters, non linear or adaptive were proposed, filters that preserves edges while suppressing impulsive noise. The non linear Median Filter has had a good performance in such cases, it replaces the input signal value at each point by the median of the signal value in a neighborhood around that point [12]. Lee and Kassam, based on a Maximun Likelihood Estimators theory, proposed an algorithm to get the median [12]. It is stated as: The output yk of the median filter M is defined as the solution of the equation :

$$\sum_{k=1}^{N} f\left(x_k - M\right) = 0 \qquad (28)$$

If f(x) is a linear function f(x)=ax, the media is obtained. If f(x) is no linear approaching the hard limiter as shown in Fig. 13, the median is obtained.

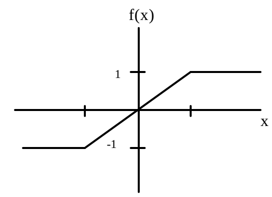

Fig. 13 Hard limiter function.

A macromodel to solve the algorithm is shown in Fig. 14, together with the simulated output in Fig.15.

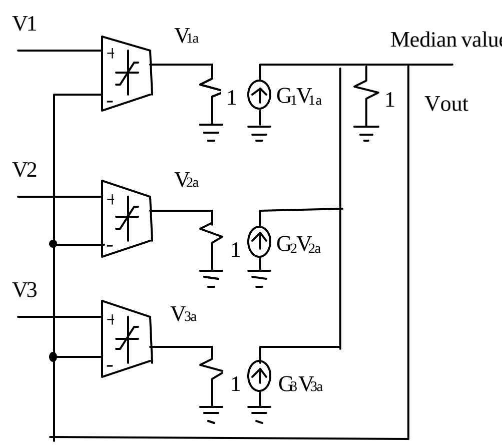

Fig.14 Macromodel to obtain the median.

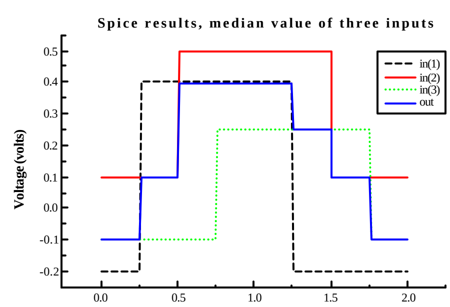

Fig 15 Median of three signals.

3 conclusions

Some classical non linear problems, a brief explanation, its mathematical model and simulation of these models using Pspice were presented in this article. Different examples from different areas were given, biochemical areas, ecological ones, an hydraulic system, a control theory example, non linear filters and a chaotic system. One can easily see the non linearities in these models, the time response of the simulated system is observed, the frequency response can also be obtained and relating two variables the state space portrait can be obtained. Simple models were used, they were simulated it with behavioral macromodels in Spice. With minor changes you can observe different responses. These models can be generalized to obtain solutions of ODE non linear systems. Graphical solutions gives a very good visualization of the problems.

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