A class of Adaptive H∞ Control Design for Linear Systems via State Feedback

Leonardo Acho Zuppa
Centro de Investigación y Desarrollo de Tecnología Digital (CITEDI-IPN)
Tijuana B.C., México
Correspondence: CITEDI-IPN 2498 Roll Dr. #757, Otay Mesa, San Diego California, 92154-7275, USA

Abstract: - In this paper a class of adaptive H∞ control design is presented for linear systems. Numerical results display that the convergence obtained of the adaptive H∞ controller of the closed loop system for the unperturbed motion is faster than the classical H∞ control design, and improve the performance of the closed loop system for the perturbed motion too.

Key-Words: - H∞ control design, adaptive control, linear systems

1 Introduction

This paper is about the design of a class of adaptive H∞ controller for linear systems. This design is based in the methodology used in [1] where a class of adaptive sub-optimal control design is presented. The adaptation law used enables to the controller to operate with fast dynamics when the states of the motion are far a way from the equilibrium point and with slow dynamics when the states of the motion are close to the equilibrium point. This adaptation in the dynamics of the controller allows to the states of the closed loop systems to converge to the equilibrium point faster than the classical H∞ control design for the unperturbed motion case, and improve the performance of the closed loop system for the perturbed motion case too.

2 Problem Formulation

Consider the linear system be described by the following differential equations

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= Cx + Du \end{aligned} \qquad (1)$$

where $x(t) \in R^n$ is the state vector, $u \in R^m$ is the control input, $w \in R^r$ is the unknown disturbance, $z \in R^l$ is the virtual output to be controlled. A, B_1, B_2, C, and D are known constant matrices with appropriate dimensions.

The following assumptions on system (1) are used hereafter,

$$A1)\ C^T D = 0$$
$$A2)\ D^T D = I.$$

A static controller of the form

$$u = K(x) \qquad (2)$$

is said to be an admissible controller if the disturbance-free ($w=0$) of the closed loop system (1), (2) is asymptotically stable.

Given a real number $\gamma \geq 0$, it is said that system (1), (2) has L_2 gain less than γ if the response z, resulting from w for initial state $x(t_o)=0$ satisfies

$$\int_{t_o}^{t_1} \|z(t)\|^2 dt \leq \gamma^2 \int_{t_o}^{t_1} \|w(t)\|^2 dt \qquad (3)$$

for all $t_1>t_o\geq 0$ and all piecewise continuous function $w(t)$.

The (classical) H∞ control problem consist to find such an admissible controller (2) that L_2 gain of the closed loop system (1), (2) is less than γ.

Theorem 1 Suppose that the systems (1) meets assumption A1) and A2). If there exist a positive definite matrix P for a given $\varepsilon>0$ and $\gamma>0$ satisfying the following Riccati equation

$$A^T P + PA + P(\frac{1}{\gamma^2} B_1 B_1^T - B_2 B_2^T)P$$

$$+ C^T C + \varepsilon I = 0 \qquad (4)$$

then the H∞ control problem is solvable and one state feedback solution is given by

$$u = -B_2^T P x(t) \quad . \qquad (5)$$

Proof.- The above result is well know and for easy of reference, this result can be obtained from [2] using the appropriate modifications.

Suppose we augment the control law [1]

$$u = -(1+\delta)B_2^T Px(t) \quad (6)$$

where δ is updated according to the following adaptive algorithm

$$\dot{\delta} = -\alpha \log(1+\delta) + \frac{1}{\log(1+\delta)+1} f(\delta, x) \quad (7)$$

where

$$f(\delta, x) = x^T[\{2(1+\delta)-1\}PB_2B_2^T P + C^T C]x \quad (8)$$

and where $\alpha>0$.

Lemma 1. Suppose the ordinary differential equation in (7) has the initial condition $\delta(t_o)= \delta_0 \geq 0$, then $\delta \geq 0$ for all $t>t_0$.

Proof. See [1].

3 Main Result

The adaptive admissible controller is the controller of the form (6), (7), (8) such that the disturbance-free closed loop system (1), (6), (7), and (8) is asymptotically stable subject to $\delta(t_o)= \delta_0 \geq 0$. The adaptive H_∞ control problem is to find an admissible controller (6), (7), (8) such that given a real number $\gamma \geq 0$, the response z, resulting from w for initial state $x(t_o)=0$ and $\delta_0 \geq 0$ satisfies

$$\int_{t_o}^{t_1} \|z(t)\|^2 dt \leq \gamma^2 \int_{t_o}^{t_1} \|w(t)\|^2 dt + \int_{t_o}^{t_1} \delta^2 \varphi(x) dt \quad (9)$$

for all $t_1>t_o\geq 0$ and all piecewise continuous function $w(t)$, and where $\varphi(.)$ is a nonnegative function. The second term on the right-hand side of equation (9) corresponds to the penalty contribution of the dynamic in (7).

Theorem 2. Suppose that system (1) satisfies conditions A1) and A2). If there exist a positive definite matrix P for a given $\varepsilon>0$ and $\gamma>0$ satisfying the Riccati equation (4), then the controller (6), (7), (8) with $\delta(t_o)= \delta_0 \geq 0$ is a solution to the adaptive H_∞ control problem.

Proof.- To begin with, let us introduce the function

$$H(x,w,u) = \frac{\partial V}{\partial x}[Ax + B_1 w + B_2 u] + \|z\|^2 - \gamma^2 \|w\|^2$$

$$= \frac{\partial V}{\partial x}[Ax + B_1 w + B_2 u] + \|Cx\|^2 + \|u\|^2 - \gamma^2 \|w\|^2 \quad (10)$$

which is quadratic in (w, u). The partial derivatives of $H=H(x,\delta, w, u)$ with respect to w, and u, yields,

$$\left.\frac{\partial H}{\partial w}\right|_{w=w^*} = \frac{\partial V}{\partial x} B_1 - 2\gamma^2 w^{*T} = 0$$

$$\left.\frac{\partial H}{\partial u}\right|_{u=u^*} = \frac{\partial V}{\partial x} B_2 + 2u^{*T} = 0.$$

Solving the above equations for w^*, and u^*

$$w^* = \frac{1}{2\gamma^2} B_1^T (\frac{\partial V}{\partial x})^T, \quad (11)$$

$$u^* = -\frac{1}{2} B_2^T (\frac{\partial V}{\partial x})^T. \quad (12)$$

$H(x, w ,u)$ can be expressed as

$$H(x,w,u) = H(x,w^*,u^*) - \gamma^2 \|w - w^*\|^2 + \|u - u^*\|^2 \quad (13)$$

Next, let us define the function

$$V(x,\delta) = x^T Px + (1+\delta)\log(1+\delta) \quad (14)$$

where

$$\frac{\partial V}{\partial x} = 2x^T P \quad (15)$$

and

$$\frac{\partial V}{\partial \delta} = 1 + \log(1+\delta) \quad (16)$$

With

$$x^T[PA + A^T P - P(B_2 B_2^T - \frac{1}{\gamma^2} B_1 B_1^T)P + C^T C + \varepsilon I]x = 0$$

we have

$$H(x,w^*,u^*) = -\varepsilon \|x\|^2$$

and equation (13) reduce to

$$H(x,w,u) = -\varepsilon \|x\|^2 - \gamma^2 \|w - w^*\|^2 + \|u - u^*\|^2$$

Using $u=(1+\delta)u*$ the above equation yields

$$H(x,w,u)=-\varepsilon\|x\|^2-\gamma^2\|w-w*\|^2+\delta^2\|u*\|^2$$
$$=-\varepsilon\|x\|^2-\gamma^2\|w-w*\|^2+\delta^2x^TPB_2B_2^TPx \quad (17)$$

From (10) and (17), the next equation is obtained

$$\frac{\partial V}{\partial x}[Ax+B_1w+B_2u]=-\varepsilon\|x\|^2-\gamma^2\|w-w*\|^2$$
$$+\delta^2x^TPB_2B_2^Tx-\|Cx\|^2-\|u\|^2+\gamma^2\|w\|^2 \quad (18)$$

or

$$\frac{\partial V}{\partial x}[Ax+B_1w+B_2u]\le-\varepsilon\|x\|^2$$
$$+\delta^2x^TPB_2B_2^TPx-\|z\|^2+\gamma^2\|w\|^2. \quad (19)$$

The time derivative of the Lyapunov function (14) is

$$\dot V=\frac{\partial V}{\partial x}[Ax+B_1w+B_2u]+\dot\delta[1+\log(1+\delta)]. \quad (20)$$

Next we are going to prove asymptotic stability for the unperturbed motion ($w=0$). With $w=0$ and using (15), (6), and (7), the equation (20) yields,

$$\dot V=x^T[PA+A^TP-2(1+\delta)PB_2B_2^TP]x$$
$$-\alpha\log(1+\delta)[1+\log(1+\delta)]+f(\delta,x) \quad (21)$$

Replacing the terms A^TP+PA in the above equation with $-P(\frac{1}{\gamma^2}B_1B_1^T-B_2B_2^T)P-C^TC-\varepsilon I$ per the ARE in (4) and using (8), equation (21) gives the result

$$\dot V=x^T[-\frac{1}{\gamma^2}PB_1B_1^TP-\varepsilon I]x$$
$$-\alpha\log(1+\delta)[1+\log(1+\delta)]$$
$$\le-\varepsilon\|x\|^2-\alpha\log(1+\delta)[1+\log(1+\delta)]$$

This proves asymptotic stability of the unperturbed motion subject to $\delta(t_o)=\delta_0\ge 0$.
Finally, to establish (9), from equation (19) we have

$$\dot V\le-\varepsilon\|x\|^2+\delta^2x^TPB_2B_2^TPx-\|z\|^2$$
$$+\gamma^2\|w\|^2\le+\delta^2x^TPB_2B_2^TPx-\|z\|^2$$
$$+\gamma^2\|w\|^2$$

which we concluded

$$\dot V\le+\delta^2x^TPB_2B_2^TPx-\|z\|^2+\gamma^2\|w\|^2. \quad (22)$$

Integration of (22) for any trajectory of the closed loop system (1), (6), (7) and (8) starting at $x(t_0)=0$ produce

$$\int_{t_o}^{t_1}(-\|z\|^2+\gamma^2\|w\|^2+\delta^2\|B_2^TPx\|^2)dt\ge$$
$$V(x(t_1))-V(x(t_0))\ge 0$$

From this last equation, we have

$$\int_{t_o}^{t_1}\|z\|^2dt\le+\gamma^2\int_{t_0}^{t_1}\|w\|^2dt+\int_{t_0}^{t_1}\delta^2\|B_2^TPx\|^2dt$$

where we obtain (9) with $\varphi(x)=\|B_2^TPx\|^2$. This concluded proof theorem 2.♥

4 Numerical example

To support the controller obtained in Theorem 1, a numerical example is develop in this section. Suppose the system in (1) takes scalar values where $A=0.5$, $B_1=B_2=1$, $C=[1\ 0]^T$, $D=[0\ 1]^T$ and $\alpha=0.5$. With $\gamma=2$ and $\varepsilon=0.1$, the positive definite solution to the Riccati equation (4) is $P=2.0491$. The simulation results for $w=0$ are shown in Fig.1, where the dotted line represents the classical H_∞ controller meanwhile the solid line represents the adaptive H_∞ controller proposed in Theorem 1. Simulation results for $w=1$ are shown in Fig. 2, where solid line is the adaptive version and dotted line is the classical one. The initial condition for (7), in both simulations, was $\delta(0)=0$.

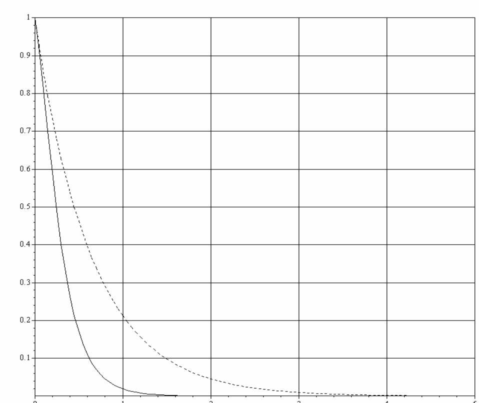

Fig.1 Plots of the trajectory ($x(t)$ versus time) for the non-disturbance case.

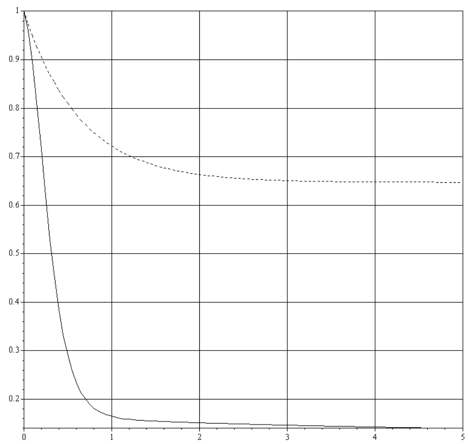

Fig. 2 Plots of the trajectory (*x(t)* versus time) for the permanent disturbance case.

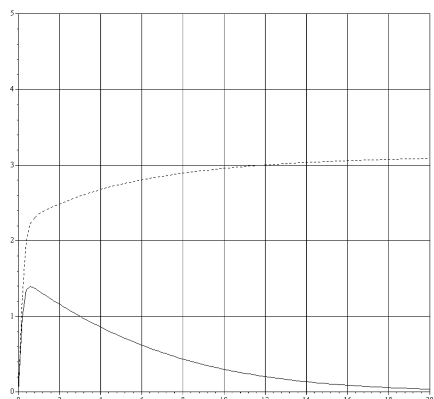

Fig. 3 Plots of δ versus time. Dotted line for the perturbed case and solid line for the un-perturbed case.

4 Conclusions

In this paper the idea presented in [1] was extended for the H_∞ control case. The result produced show that the adaptive H_∞ controller, besides to have faster convergence for the un-perturbed case, it has more robustness for the perturbed case than the classical H_∞ controller.

References:

[1] Hench J. J., On a class of adaptive sub-optimal Riccati-based controller, *Proceeding of the American Control Conference*, 1999, San Diego, California, USA, pp. 53-55.

[2] Jian-Bo He, Qing-Guo Wang, and Tong-Heng Lee. H_∞ disturbance attenuation for state delayed systems, *Systems and Control Letters*, Vol. 33, 1998, pp. 105-114.