Gravitational Waves Detection via Sequential Analysis

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Abstract: - The problem of Gravitational Waves (GW) detection is dealt with from a signal processing perspective. Our aim is to present the two main classes of GW waveforms which are of practical relevance, and some of the open problems concerning their detection. In fact, for both of them, the classical Generalized Likelihood Ratio Test (GLRT) is not practically feasible because of the associated computational demands. For the class of signals known as pulsar GW, we present some recent promising detection approaches which are based on sequential procedures.

Key-words: - Gravitational waves, data analysis, sequential tests.

1 Introduction
The detection of Gravitational Waves (GW) is currently one of the most challenging research topics. There are many GW detection projects which are active around the world, as LIGO (USA), VIRGO (Italy-France), GEO600 (Germany) and TAMA (Japan).

In fact, the construction of GW antennas has required huge economical and human investments, which have been justified by the extraordinary impact that the direct observation of GW may have on our understanding of the Universe. Physicists ensure that the systematic observation of gravitational radiation may disclose a new window for studying the Universe around us and the forces that rule it. This is the future scenario that can be foreseen, and the effective possibility of reaching it strongly relies on the ability to bring together different scientific and technical competencies.

Engineering and experimental physicists gave their fundamental contribution in building one of the most sophisticated and technologically advanced scientific instrument: the GW antenna. Presently, many of these instruments are in a phase of calibration or testing and, in some cases, are collecting the first useful data runs. Thus the contribution of the signal processing community is becoming of fundamental importance in analyzing the data, for detection and parameters estimation purposes.

For these reasons it is expected that GW detection will become a hot topic among the signal processing community, and it is desirable that the researchers involved in this field be stimulated to give their contribution. In effect, almost all of the GW signal processing oriented papers have appeared on physics journals, so that the diffusion of the relevant problems among the signal processing community has been rather limited.

2 Chirp signal
This section is simply aimed to present some of the open problems, and is by no means neither original nor exhaustive.

In its simplest version the GW signal waveform known as chirp can be cast in the form of an AM/FM modulated signal:

\[ s(t) = IA(t;t_0, \tau) \cos[\beta(t;t_0, \tau) + \phi] \]

with \( t_0 \leq t < t_0 + \tau \). Here \( t_0 \) is the arrival time and \( \tau \) is the so called sweep time.

The modulation terms are

\[ A(t;t_0, \tau) = \frac{1}{\sigma_0^2} \left( 1 - \frac{t - t_0}{\tau} \right)^{-3/4} \]

The problem of Gravitational Waves (GW) detection is dealt with from a signal processing perspective. Our aim is to present the two main classes of GW waveforms which are of practical relevance, and some of the open problems concerning their detection. In fact, for both of them, the classical Generalized Likelihood Ratio Test (GLRT) is not practically feasible because of the associated computational demands. For the class of signals known as pulsar GW, we present some recent promising detection approaches which are based on sequential procedures.
and

$$\beta(t; t_0, \tau) = \frac{16}{5} f_0^3 \tau \left[ 1 - \left(1 - \frac{t - t_0}{\tau} \right) \right]^{5/8},$$

where $f_0$ is the lower cutoff frequency of the GW detector. The range of admissible values for $\tau$, which depends also upon $f_0$, is very large and this contributes to make the signal detection particularly difficult.

Based on the data $x(t)$ collected by the antenna, the chirp detection problem may be cast in the usual form of a statistical hypothesis test

$$H_0 : x(t) = w(t);$$

$$H_1 : x(t) = w(t) + s(t),$$

where $w(t)$ is a Gaussian noise term with known PSD. The unknown parameters of the signal are $I$, $\phi$, $t_0$ and $\tau$. The latter two are of physical interest and should be the objects of estimation procedures. The typical values of the signal amplitude $I$, compared with the minimum achievable noise strength, allow us to classify the detection problem as a very weak signal detection problem.

Moreover, the above is a simplified mathematical model of the actual physical problem. In fact, a more precise formulation would require a larger number of typical signal parameters, some uncertainty on the noise PSD, deviation from the Gaussianity of the noise statistics, and harmonically related sinusoidal disturbances to be ascribed to non linearity in the detection chain. Without including these, or other complications, we now describe the main reasons why the chirp detection is, from a practical point of view, a problem still unresolved.

To further simplify the exposition, let us restrict our discussion to the AWGN model in which the noise PSD is a constant, say $N_0/2$. Then, assuming the above standard formulation, it is easily seen that the GLRT detector is based on the comparison of the following statistic

$$\max_{t_0, \tau} \left| \int_0^T x(t) \frac{A(t; t_0, \tau)}{\sqrt{N_0 / f_0^4 \tau}} \exp[-j\beta(t; t_0, \tau)] dt \right|^2,$$

with a proper threshold level. Here $T$ is the observation interval.

Unfortunately, computation of the maximum with respect to $\tau$ requires the implementation of a bank of matched filters, each tuned to a specific $\tau$ value. The number of filters in the bank is so large that no practical implementation seems achievable (especially for more sophisticated and accurate signal models). In fact, evaluation of the relevant ambiguity function, $F(\Delta t_0, \Delta \tau)$, defined as

$$\int_{t_0}^{t_0+\tau_1} s(t; t_0, \tau_2) \frac{A(t; t_0, \tau_1)}{\sqrt{N_0 / f_0^4 \tau_1}} \exp[-j\beta(t; t_0, \tau_1)] dt,$$

reveals that $F(\Delta t_0, \Delta \tau)$ is a very sharp function. As a consequence, the filter bank must be designed in a very dense way, i.e., two adjacent filters should differ for very small values of $\Delta t_0$ and $\Delta \tau$. Needless to say, the described GLRT approach automatically provide the estimates of $t_0$ and $\tau$.

Then the question is: does there exist some valid alternative to the classical GLRT approach which guarantees a reasonable level of performance? The lines of investigations followed in the literature include:

1. Quadratic detection [1];
2. Sequential methods, including Page's test and quickest detection [1];
3. Hierarchical detection structures [2];
4. Hardware based efficient statistics computation [3].

Even though many interesting alternatives have been found and studied, it seems, as we have stressed previously, that there is certainly need of further investigations. The interested reader may refer to [9] to get the state of the art on this matter.

### 3 Pulsar signal

Some astrophysical sources, far from the coalescence stage that generates chirp waveform, radiate almost periodic gravitational waveforms. These are called continuous gravitational waves or, more briefly (but less properly), pulsars. Differently from the chirp case, the pulsar signal is virtually unbounded in time. This feature suggests the exploitation of sequential detection procedures that may collect the signal energy along very large time intervals, while complying with the essential requirement of moderate computational demands per unit of time. In fact, the classical approaches, in all of the various forms that have been conceived
require data FFTs to be computed on extremely large data sets. Conversely, by means of sequential data processing, the massive overall computational burden is efficiently partitioned. In other words, a relatively small amount of computation is performed at any sampling interval, and the overall observation period (i.e., the total number of sampling intervals required by the test to come at stop) is not fixed in advance, but it is allowed to be as large as it is needed in order to comply with the desired detection performances.

To be more specific, let us start by introducing the pulsar waveform. Once again we resort to an oversimplified signal model which should facilitate the understanding of the main detection issues, avoiding formal complications.

The pulsar waveform we consider, cast in a discrete-time form, is

\[ s(n) = A \sin[2\pi \nu_0 n + \beta \sin(2\pi \nu_1 n + \phi) + \vartheta] \]

where \( \nu_1 \) is the daily rotation frequency of the Earth divided by the sampling frequency, \( \beta \) is a FM modulation index and \( \nu_0 \) is the unknown normalized frequency of the gravitational radiation.

The pulsar detection problem is then formalized as

\[
H_0 : x(n) = w(n) + s(n)
\]

\[
H_1 : x(n) = w(n)
\]

where \( w(n) \) represents a white (for the sake of simplicity) and Gaussian noise term.

Basically all the classical approaches that have been proposed rely on FFT-based detection statistics, of the form

\[
\max_{\nu_0} \sum_{n=1}^{N} x(n) \exp[-j\beta \sin(2\pi \nu_1 n + \phi) - j2\pi \nu_0 n]^2
\]

where \( N \) should be as large as possible in order to collect the maximum allowable signal energy. As said before, this is a drawback of the classical approaches, which are based on fixed sample size tests.

In a different approach, one could take advantage by sequential tests, such as the well known SPRT [5]:

\[
\sum_{n=1}^{m} l[x(i)] \begin{cases} 
\geq \gamma_1 \Rightarrow \text{choose } H_1 & ; \\
\leq \gamma_0 \Rightarrow \text{choose } H_0 & ; \\
\text{otherwise } \Rightarrow \text{take another sample} \end{cases}
\]

where \( l[x(i)] \) denotes the log-likelihood ratio pertaining to the data sample \( x(i) \). As it should be clear, the total number of samples \( N \) which will be required is not fixed in advance, but depends on the particular realization of the data sequence. The average value of the random variable \( N \) is one of the performance figure of the SPRT. Other important figures are the detection probability \( P_d \) and the false alarm probability \( P_f \). They are related to the threshold levels \( \gamma_1 \) and \( \gamma_0 \) by the simple, though approximate, formulas:

\[
\gamma_1 \approx \log \left( \frac{P_d}{P_f} \right) ; \\
\gamma_0 \approx \log \left( \frac{1 - P_d}{1 - P_f} \right)
\]

Actually, the SPRT, at least in the formulation presented above, requires that the data are iid under both the hypotheses, and this is certainly not true in our case, just because of the presence of the time-varying pulsar signal \( s(n) \) to be detected.

However, by properly reformulating the problem in the frequency domain, the potential of the SPRT can be fully exploited. Let us consider data frames of length \( L \). Such data dimension is chosen in such a way that the doppler term appearing in the pulsar waveform, i.e. \( \beta \sin(2\pi \nu_1 + \phi) \), can be considered approximately constant. Each data frame is then transformed in the frequency domain, and the sample at frequency \( \nu_0 \), say \( X_k(\nu_0) \), is retained (Fig.1). The subscript \( k \) reminds that the frequency sample refers to the \( k \)th data frame. Then the constant sequence \( X_k(\nu_0) \) plus additive Gaussian noise, \( k=1,2,\ldots \), is tested against the only noise hypothesis. Cast in this form the problem can be regarded as a constant signal detection embedded in AWGN. Accordingly, the sequential procedure outlined above can be immediately pursued.

Due to the presence of unknown amplitude and phase, we must employ an incoherent approach which results in retaining the square modulus \( |X_k(\nu_0)|^2 \) of the collected sample. We would like to stress that such proposed approach is inspired by previously published works, see [6] and [7], where the detection of a sine wave with undesired phase drift was addressed. The detection strategy there proposed is referred to as multi-incoherent
With respect to those papers, we further propose the idea of processing the data in a sequential fashion, which is, we feel, particularly suited for the problem at hand.

As the GW signal frequency $\nu_0$ is unknown to the receiver, the outlined procedure is to be replicated for all the values of frequency of interest. In practice, this task can be accomplished by implementing a bank of sequential detectors, where each branch in the bank tests a value of frequency by computing the correspondent test statistic and comparing it, at each sampling interval, with two threshold levels. The final decision is taken according to the following rules:

- a decision in favor of $H_1$ is taken at the moment that any branch in the bank crosses the correspondent upper threshold; in this case the detection procedure ends.
- a decision in favor of $H_0$ is taken if all of the detection statistics fall at least once below their lower threshold levels.

Beside the frequency, the signal amplitude $A$ is also an unknown parameter. Circumventing the signal amplitude uncertainty may require to fix a minimum allowable value for $A$, say $A_m$, according to which the detector is designed. Namely, the detection and the false alarm probabilities are chosen by assuming that the signal amplitude is just $A_m$. It can be shown that larger values of $A$ bring to performance improvements. Thus the detector design is conservative in the sense that the actual performances are not less that the nominal ones assumed at design stage. A more detailed dissertation, which includes analytical approximations for the detection performances and simulation studies, is in preparation [8]. Here we offer some of the obtained approximated formulas for the average value $N$ of the number of data frames required to end the test, under both the hypotheses, assuming $\nu_0$ known.

By defining a signal-to-noise ratio $\lambda$ for single data frame, $\lambda=A^2L/(2\sigma^2)$, it can be shown that

$$E[N \mid H_1] = \frac{P_1\gamma_1 + (1-P_1)\gamma_0}{\lambda};$$

$$E[N \mid H_0] = \frac{P_1\gamma_1 + (1-P_1)\gamma_0}{\lambda};$$

where $I_0(y)$ is the modified Bessel function of zero order. A particularly easy-to-handle version of the above relationships can be obtained in the limit of vanishingly small $\lambda$ which, we remind, is certainly the typical scenario when we deal with GW detection problems.

In fact, straightforward algebra yields:

$$E[N \mid H_1] = \frac{P_1\gamma_1 + (1-P_1)\gamma_0}{\lambda};$$

$$E[N \mid H_0] = \frac{P_1\gamma_1 + (1-P_1)\gamma_0}{\lambda};$$

which can be used for a first evaluation of the detector performances.

By using the previous formulas, it can be shown that the sequential approach is less computationally demanding with respect to an equivalent incoherent fixed sample procedure, as long as the average value of the number of data frames to be processed is concerned. For instance, in Figure 2 the value of $E[N]$ is plotted versus the detection probability, for a given value of false alarm rate, for the proposed sequential test (continuous line) and for the classical one (dashed line). In the latter case, obviously, the value of $N$ is deterministic and the statistical expectation is inessential. The provided plot refers to $H_1$. Similar general considerations hold for the null hypothesis $H_0$.

4 Conclusion

The direct observation of Gravitational Waves (GW), is one of the most challenging scientific enterprises of modern physics. Hugely
Fig. 2  A comparison between the average number N (hypothesis H1), for a multi-incoherent sequential procedure (continuous line), and a multi-incoherent fixed test (dashed line).

sophisticated instruments, designed to record the tiny distance variation (in the order of one part over $10^{21}$) between several test masses caused by the impact of a GW, have been conceived in the last decades. Many of them, in particular those based on interferometric principles, are collecting the first useful data runs.

In this scenario it is expected that an increasingly large number of research groups around the world may focus on the design of detection and estimation procedures, specifically tailored to the GW detection problem.

In this paper we have tried to introduce the reader to a few open problems in this field, obviously without any pretence of completeness. In addition, some new ideas about sequential detection procedures for the detection of periodic GW have been presented. Our current investigations include:

1) Alternative to the GLRT for chirp signals.
2) Detailed design of sequential detector for pulsars.

References: