

# Sidelobe Minimisation in Integer-Positioned Sparse Antenna Arrays

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*Abstract:* – Modern array signal processing applications place great demands on the underlying antenna array geometry to deliver high performance. This paper discusses a computational solution to an integer multicriteria optimisation (MO) problem that arises in a certain class of nonuniform linear array (NLA) design. “Fully augmentable” arrays have a full and complete set of covariance lag differences; within the class of fully augmentable geometries, we seek an  $M$ -sensor layout that yields the smallest sidelobes in the array beam pattern, possibly only within a particular angular range. The proposed method permits near-optimal minimisation in a feasible amount of computer time.

*Key-Words:* – Antenna arrays, sparse array geometry, integer optimisation, sidelobe minimisation.

## 1 Introduction and Background

The history of major developments in antenna array geometry design is, to a great extent, the history of achievements in numerical optimisation. As early as 1946, the pioneering contribution of Dolf [1] to array design was already formulated as the solution of an optimisation problem. Following the appearance of this and the fundamental paper of Woodward and Lawson [2] in 1948, the array geometry design problem was firmly established as an optimisation problem, and each successive breakthrough in mathematical programming has been closely followed by its associated application to array design.

During these early years, only single conventional antennas (“dishes”) were implemented in practice, and hence optimisation efforts were at first concentrated upon the current distribution function, defined by the desired antenna beam pattern. Later, when the first antenna arrays were assembled, it was immediately realised that the distribution of individual sensors along a line (linear arrays) or across some area (planar arrays) was crucial to the optimisation process as well.

The first attempts to optimise antenna array geometry considered conventional beamforming. While these problems were formulated in terms of integer programming, later developments in dynamic programming were then used to find solutions, such as in the pioneering work of Skolnik, Nemhauser and Sherman [3], and Lo and Lee [4].

For the group of scientists involved mainly with radar and communications applications, minimum antenna array beam pattern sidelobe level was a ma-

ior goal in these optimisation studies (see Dooley [5], for example). Meanwhile radio-astronomers tackled NLA design from a completely different viewpoint: based on interferometry principles, they noticed that “repetition of the same spacing [between pairs of array sensors] gives redundant information, which is generally useless” (Biraud, Blum and Ribes [6]).

After the important conjecture made by Arsac [7, 8], who may have been the first engineer to introduce the unique optimal-lag four-sensor geometry  $\mathbf{d} = [0, 1, 4, 6]$ , Barber [9], Haubrich [10], and mainly Moffet [11] elaborated on Arsac’s work, incorporating certain results from number theory (eg. Leech [12]), in turn proposing various optimisation techniques in order to obtain larger arrays. These papers properly formulated the integer programming problem for nonredundant and minimum-redundancy antenna array design, though strictly optimal solutions were obtained only by exhaustive search, despite numerous studies.

## 2 Problem Formulation

In many direction-finding applications, the number of antenna sensors available for the construction of an array is limited, in which case the problem of optimum antenna geometry for a fixed number of sensors  $M$  naturally arises. For linear arrays, solutions to this problem belong to the class of nonuniformly spaced linear arrays (NLAs), also known as *sparse* or *aperiodic* arrays; several different approaches currently exist to determine the “best” design, for exam-

ple [13].

Meanwhile, most speculation regarding optimum sparse geometry has concerned direction-finding applications involving the independent (Gaussian) source model. For example, the well-known suggestion that a minimum-redundancy criterion is appropriate for (integer-spaced) NLA geometry optimisation [14] is based on the simple fact that such geometries generate a complete (“gapless”) set of spatial covariance lags. For independent Gaussian sources, this property immediately allows the unambiguous estimation of up to

$$m < \frac{1}{2}M(M-1) \quad (1)$$

directions-of-arrival by the direct augmentation approach (DAA) of Pillai *et al.* [15].

Suppose that we have  $M$  antenna sensors and wish to construct the “optimal” NLA geometry. In this context, “optimal” means that the resulting array beam pattern has minimal sidelobes; nevertheless, it will become apparent that our optimisation technique can easily operate with a different optimisation criterion (for example, see [16]). Let the array sensor positions

$$\mathbf{d} \equiv [d_1 \equiv 0, d_2, d_3, \dots, d_M \equiv M_\alpha - 1] \quad (2)$$

be restricted to integer values (usually measured in half-wavelength units). Fully augmentable arrays have the property that the  $M(M-1)/2$ -variate set of all intersensor distances

$$\mathcal{D} \equiv \{d_j - d_k \mid j, k = 1, \dots, M; j \geq k\} \quad (3)$$

is complete, *ie.*  $\mathcal{D} = \{0, \dots, d_M\}$ . Recall that the *co-array*  $\mathbf{c}$  of a linear array  $\mathbf{d}$  is the sorted set of nonduplicated elements of  $\mathcal{D}$ , thus the  $M_\alpha$ -element co-array corresponding to every fully augmentable NLA is uniform.

Optimum-lag NLAs have a complete *and* nonredundant set of covariance lags. Since these exist only in a very limited number of cases ( $M \leq 4$ ) [11], some of the covariance lags present in the difference set of a fully augmentable array are generally duplicated, and this redundancy (in the form of the number of redundant lags) is often minimised. The class of sparse arrays introduced by Moffet [11] to achieve maximum resolution for a given number of antenna sensors are known as “minimum-redundancy” arrays. Further classes are defined and their properties discussed in [17].

A typical example of a five-sensor fully augmentable NLA is

$$\mathbf{d}_5 = [0, 2, 5, 8, 9] \quad (4)$$

with co-array  $\mathbf{c}_5 = [0, 1, \dots, 9]$ . By symmetry, we also have that  $\mathbf{d}_5 \equiv [0, 1, 4, 7, 9]$ .

We constrain ourselves to the class of fully augmentable arrays, since their complete co-arrays perfectly suit interferometric or covariance moment-based spectral (direction-of-arrival) estimation methods.

The antenna array beam pattern is defined as

$$f(\theta) = \left| \sum_{k=1}^M \mathbf{w} \exp \left[ -i 2\pi \frac{\delta}{\lambda} d_k \sin \theta \right] \right|^2 \quad (5)$$

where  $\mathbf{w}$  is an optional weighting function (such as the Blackman window),  $\delta$  is the intersensor distance, and  $\theta \in [-\pi/2, \pi/2]$ .

We suggest that it is useful at this stage to introduce the parameters  $\theta_{\min}$  and  $\theta_{\max}$ , and seek minimum sidelobe levels only within the (possibly restricted) angular range  $\theta \in [\theta_{\min}, \theta_{\max}]$ .

### 3 Proposed Three-Stage Solution

Thus the statement of the optimisation problem is that we are to distribute a given number of sensors  $M$  over the 1-D grid of integer values to create a fully augmentable NLA, in the way that achieves the least maximum sidelobe level within the angular range  $\theta \in [\theta_{\min}, \theta_{\max}]$ . The optimisation could be written in terms of the cost function

$$\min_{\mathbf{d} \in \Phi} A(\mathbf{d}, \mathbf{w}, \theta_{\min}, \theta_{\max}) \quad (6)$$

where  $\Phi$  represents the class of fully augmentable NLAs. Physically, we are attempting to balance the competing criteria of sparsity against total antenna array aperture, all the while seeking minimal sidelobes. Hence we are dealing with a constrained integer multicriteria (or multi-objective) optimisation (MO) problem [18, 19].

It should be clear that the non-trivial computational complexity of the cost function in conjunction with the “combinatorial explosion” when considering the total number of different possible geometries  $\mathbf{d}$  immediately rules out the practical possibility of a brute force (exhaustive global search) solution to this problem. Indeed, since effective computational algorithms for finding the strictly optimal solution have not yet been developed even for less sophisticated integer programming problems such as minimum-redundancy optimisation *per se*, we have developed the following *ad hoc* three-stage optimisation approach to find a close-to-optimal solution.

In the first stage, we define the total array aperture by selecting a nonredundant array geometry of

$M_1$  elements, for example from the lists published in [20, 21]. A nonredundant array has maximum aperture for a given number of covariance gaps. (As mentioned above, any nonredundant array with more than four elements must have gaps.) At this stage we have  $(M - M_1)$  elements remaining at our disposal to distribute optimally amongst the vacant integer positions.

Secondly, we eliminate all gaps with the minimum possible number of elements  $M_2$ , using an exhaustive tree search technique. (Since this stage does not refer to the cost function, it is computationally fast.) This results in an (often lengthy) list of gap-free (fully augmentable) candidate arrays.

Thirdly, we utilise the remaining  $M_3 = (M - M_1 - M_2)$  degrees of freedom by adding this number of elements to each candidate geometry in turn, searching for the array that minimises the selection criterion (cost function)  $A(f)$ . For each candidate, we conduct an integer programming search by adding elements to vacant positions one-by-one, at each stage simply selecting the maximum- $A$  geometry for further investigation. This method will find the global optimum (of this third stage) provided that the problem is separable, *ie.* that each of the  $M_3$  degrees of freedom is independent of the other; while this is not strictly true, the small level of interaction between successive introduced elements relegates this to a second-order effect. This third stage of our approach is similar to the dynamic programming schemes used in [3, 4].

By itself, this would be in principle a theoretically straight-forward optimisation problem, since for a given  $M$ ,  $\mathbf{w}$ ,  $\theta_{\min}$  and  $\theta_{\max}$ , we can compute the  $A$ -optimal NLA. However, the entire three-stage optimisation problem should now be wrapped inside an  $M_1$ -optimisation (by exhaustive search), since our initial choice of  $M_1$  was quite arbitrary.

## 4 Results and Discussion

The following example illustrates array geometry optimisation results for an  $M = 16$  sensor array, with no beam pattern weighting and no restrictions on angles of interest, *ie.*  $\theta_{\min} = -\pi/2$  and  $\theta_{\max} = \pi/2$ . In the first stage, the choice  $M_1 = 10$  immediately gives us the initial 10-sensor nonredundant Sverdlik array (see [20] or Appendix A of [22]):

$$\mathbf{d}_{55}^{(0)} = [0, 1, 6, 10, 23, 26, 34, 41, 53, 55]. \quad (7)$$

The exhaustive tree search of stage two yields 37 candidate fully augmentable geometries, each with 14 sensors and 36 redundancies.

The integer programming minimisation of stage three finds that of these 37 candidates,

$$\mathbf{d}_{55}^{(1)} = [0, 1, 6, \underline{9}, 10, 23, 26, 34, \underline{37}, 41, \underline{48}, \underline{51}, 53, 55] \quad (8)$$

is best, since with the addition of two sensors it yields the desired 16-sensor NLA

$$\mathbf{d}_{55} = [0, 1, 6, \underline{9}, 10, \underline{22}, 23, 26, \underline{32}, 34, \underline{37}, 41, \underline{48}, \underline{51}, 53, 55] \quad (9)$$

that has a maximum sidelobe level of  $-9.06$  dB relative to the central beam. Thus we have partitioned our  $M = 16$  sensors in this example in the form  $\{M_1 = 10, M_2 = 4, M_3 = 2\}$ .

Note that this three-stage optimisation search was completed utilising only a few minutes of computing time on a modern workstation.

By way of comparison, Fig. 1 shows the distribution of maximum sidelobe levels for 5000 randomly constructed integer NLA geometries with the same constraints as  $\mathbf{d}_{55}$ . The minimum sidelobe level obtained in this Monte-Carlo simulation (that used more than twice as much computer time as our three-stage optimisation) was  $-9.09$  dB. Evidently, our optimisation approach does indeed find the close-to-optimal solution. Moreover, as  $M$  increases to more realistic values, the saving in computer time becomes proportionally much greater.

Fig. 2 shows the optimisation results for the situation where initial arrays are defined by the Sverdlik sequences  $M_1 = 10, 9, 8, 7$  and sidelobe minimisation is desired over the angular range  $[-10^\circ, 10^\circ]$ . We see that the resulting minimisation improves as we decrease  $M_1$ , however at the expense of a broadening central beam. This can be understood in terms of increasing degrees of freedom (as  $M_1$  decreases) in the second and third optimisation stages.

## 5 Summary and Conclusions

We have introduced a sidelobe-minimisation problem involving constrained nonuniformly spaced antenna array geometry optimisation. This difficult optimisation problem is formulated in terms of a multicriteria integer programming problem, where general computational schemes to efficiently find the globally optimum solution are unknown. Therefore the optimisation problem is reduced to a simplified form where effective techniques can be applied, based on integer programming principles.

The three-stage optimisation procedure for finding a close-to-optimal solution may be applied to other

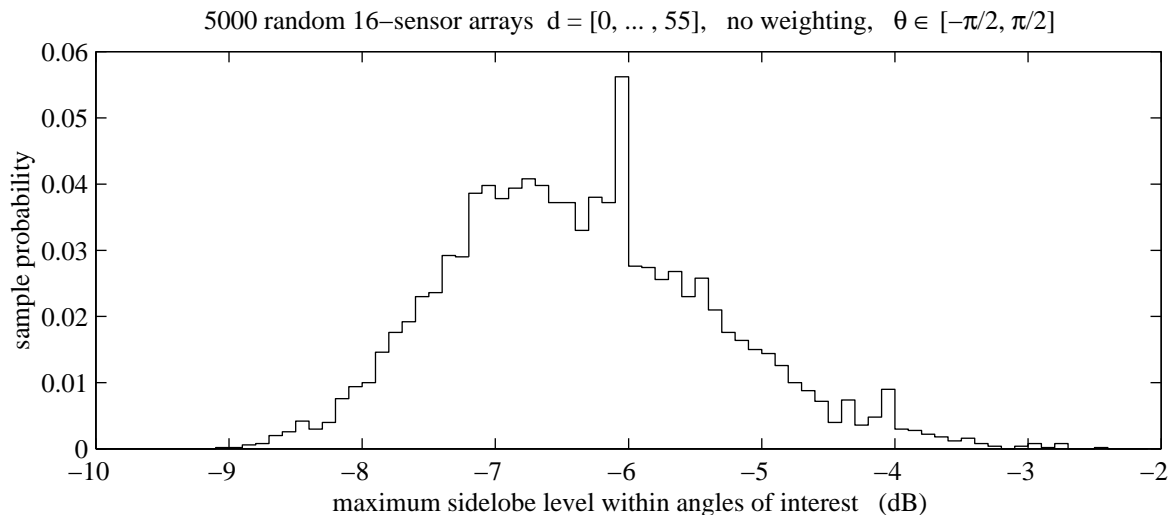


Figure 1: Distribution of maximum sidelobe levels for random arrays competing with  $d_{55}$ .

antenna array geometry design problems of a similar nature.

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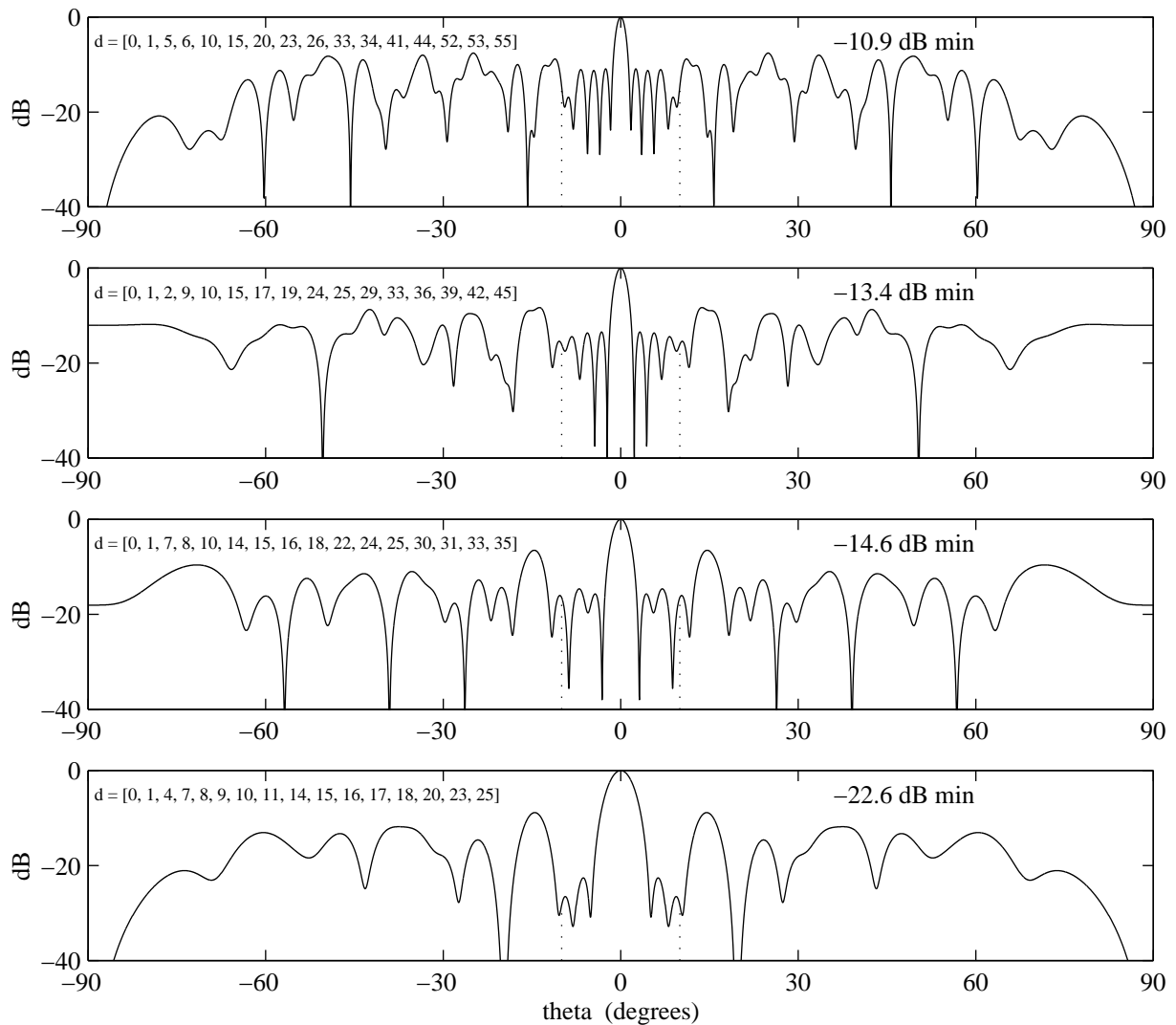


Figure 2: Sidelobe optimisation results for the angular range  $[-10^\circ, 10^\circ]$  for initial NLAs of  $M_1 = 10, 9, 8, 7$  (from top to bottom, respectively).

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