Correlation Properties Of Binary Sequences Generated By The Logistic Map-Application To DS-CDMA

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Abstract — In this paper, we will show that a simple one dimension nonlinear map allows generating binary sequences that have better correlation properties then classical pseudo random ones. Using performance criterion that is suitable for CDMA application, the performances of proposed binary sequences will be compared to the performances of Gold ones. This comparison will show that nonlinear systems generate sequences that allow a signal to noise ratio better then pseudo-random sequence generated by linear shift registers.

Keywords — Correlation properties, Kneading sequences, DS-CDMA, Spreading sequences.

I. INTRODUCTION

It is well known that the choice of spreading sequences is very important in the conception of a CDMA system. In the case where the symbols to be transmitted are binary, it was shown [1-2] that aperiodic, even and odd correlation function have the same importance in mitigating multi user interference, on the other hand, sequences with good periodic auto correlation function allow fast code synchronisation. But, in general, it is difficult to find a tradeoff in resolving these problems and every classical type of sequences presents a gap in at least one type of correlation function, so works are still conducted to find better types of sequences. It was shown in [4] that non-linear map could generate sequences able to concurrence classical ones. Moreover, truncated then quantized chaotic sequences generated by the tent map were considered and compared numerically to Gold and m-sequences and it was found that in many cases chaotic sequences behave better. The general problem for those sequences is to find initial conditions that allow good statistical properties. This is due to the fact that chaotic sequences are sensitive to initial conditions, two sequences beginning by two very close points are in general topological and statistical different. In this contribution, we will consider kneading sequences generated by the logistic map. The advantage of these sequences is that they correspond to super stable orbits that are not sensitive to initial conditions, and so are easy to generate, more over every binary sequence can be generated using a corresponding super-stable orbit. In [8], it was shown that kneading sequences allow better performances than Gold-like sequences for the case of period 15. The considered period in this paper is 31, we will show that kneading sequences existing with a great number, allow various correlation properties. As an application we will consider the DS-CDMA and kneading sequences will be compared to Gold ones. For the comparison, we will use the criterion LSE/AO used in [3,7] to select individually good sequences, then among the selected sequences we will keep only that are less correlated. The paper is organised as following, in the first section, we will present the DS-CDMA system and the performance criterion that we will use. In the second section, we will introduce the notion of symbolic sequences generated by a non-linear system and some of their properties, we will present also the various correlation properties allowed by this sequences. In the third section we will give a performance comparison of the kneading and Gold sequence of period 31. We conclude and give some perspectives in the last section.

II. SEQUENCE PERFORMANCES IN DS-CDMA SYSTEM

In figure 1 is represented the model of a Direct Sequence – Code Division Multiple Access (DS-CDMA) system.

For every user \( k \) \( b_k \) is the information signal and \( a_k \) is the spread spectrum code; and \( p_k \) is the carrier. All users receive the same signal \( r(t) \) that is the somma-
ation of all transmitted signal \( r_k(t) \) and the noise \( n(t) \).

\[
b_k = [\ldots b_k(-1)b_k(0)b_k(1)\ldots]
\]

\[
p_k = \sqrt{2P \exp(j(w_c t + \theta_0))}
\]

\( P \) is the commun signal power of all users, it is the case of the forward link, i.e base station to mobiles or the case of perfect power control in the reverse link. Let spreading sequences be of period \( N \) and denoted

\[
a_k = a_k[0]a_k[N-1]a_k[0]\ldots, k = 1, 2, \ldots, M
\]

The Aperiodic, Even and Odd Cross Correlation of two sequences \( a_k \) and \( a_i \) are defined respectively by

\[
C_{k,i}(l) = \begin{cases} 
N^{-l-1} \sum_{j=0}^{l} a_k[j]a_i[j]^* & \text{if } 0 \leq l \leq N-1 \\
N^{l+1-1} \sum_{j=0}^{l} a_k[j-1]a_i[j]^* & \text{if } 1-N \leq l \leq -1 \\
0 & \text{otherwise}
\end{cases}
\]

\[
\theta_{k,i} = C_{k,i}(l) + C_{k,i}(l-N)
\]

\[
\hat{\theta}_{k,i} = C_{k,i}(l) - C_{k,i}(l-N)
\]

The Aperiodic, Even and Odd Auto Correlation of a sequences \( a_k \) are defined respectively by

\[
C_k(l) = C_{k,k}(l).
\]

\[
\theta_k(l) = \theta_{k,k}(l)
\]

\[
\hat{\theta}_k(l) = \hat{\theta}_{k,k}(l)
\]

The channel is supposed to be an Additive White Gaussian Process with two sided spectral density \( \frac{N_0}{2} \); the bit energy is \( E_b \). In the case of a matched filter receiver (fig. 2), it has been shown in [1], that for a reference user i, the output of the matched filter required to decode the symbol number "0" \( b[0] \) is given by

\[
Z_i = \sqrt{\frac{P}{2}} T_k \left\{ b_i[0] + \sum_{k=1}^{K} \sum_{k \neq i} I_{k,i}(b^k, \tau_k, \phi_k) \right\} + N(t)
\]

\[
I_{k,i}(b^k, \tau_k, \phi_k) = \frac{1}{N} b_k[1] \left\{ C_{k,i}(l) + \gamma_k C_{k,i}(l-N) \right\} e^{j\phi_k}
\]

\[
\gamma_k = \frac{b_k[0]}{b_k[1]}
\]

The signal to noise ratio is given by

\[
SNR_i = \left[ \frac{N_0}{2E_b} + \frac{1}{6N^3} \sum_{k=1}^{M} \sum_{k \neq i}^{M} [2\mu_{k,i}(0) + \mu_{k,i}(15)] \right]^{-1}
\]

\[
\mu_{k,i}(n) = \sum_{l=-N}^{N-1} C_{k,i}(l)C_{k,i}(l+n)
\]

The \( SNR \) can also be expressed using the Auto Correlation Function [1-2] via the formula

\[
r(t) \exp[-j(\theta_i l + \phi_i)] a_k(t)^*
\]

Fig. 2. Matched Filter Receiver.

From the equations above, we can see how much even, odd and aperiodic (cross and auto) correlation functions are important in mitigating multi user interference phenomena. However, it is well known that in general, sequences with good auto correlation functions don’t have good cross correlation functions and inversely. This makes the task of selecting a spread spectrum sequence set difficult. Moreover the selection of a sequence set has to be done according to the channel characteristics and the type of transmission. The task is more complicated in real cases.

### III. KNEADING SEQUENCE GENERATOR

#### A. Kneading Sequences

The map used to generate symbolic sequences is the logistic one:

\[
f_\alpha(x) = 1 - ax^2, 0 \leq a \leq 2
\]
Let \( X = x_0x_1...x_n... \) be a sequence generated by the logistic map, i.e. \( x_0 \) is the initial condition and \( x_{k+1} = f_a(x_k) \); to \( X \) corresponds a symbolic sequence: \( S = S_0S_1...S_n... \).

\[
\begin{align*}
S_i &= 1 \text{ if } x_i > 0 \\
S_i &= -1 \text{ if } x_i \leq 0 
\end{align*}
\]

(13)

It is clear that if the sequence \( X \) is periodic with a period equal to \( N \) it is the same thing for the corresponding symbolic sequence. In this case, the Floquet multiplier of \( X \) is

\[ \lambda(X) = \frac{df_a^N}{dx} (x_0) = \prod_{k=0}^{N-1} f_a(x_k) \]

We have the following three possible situations:

- \( |\lambda(X)| < 1 \), the orbit \( X \) is stable
- \( |\lambda(X)| > 1 \), the orbit \( X \) is unstable
- \( |\lambda(X)| = 1 \), the orbit \( X \) can be stable or unstable

The Floquet multiplier of an orbit beginning by the initial condition \( x_0 \) equal to \( f_a(0) = 1 \) is null, such orbit is said to be super stable. Every symbolic sequence corresponding to a super stable orbit is called kneading sequence.

**B. Kneading Sequence Generator**

A symbolic sequence generator is represented in figure 3, it is composed of two parts, the nonlinear part followed by a quantizer. The characteristics of the generated symbolic sequence \( S_0S_1S_2... \) is defined by the parameter \( a \) of the logistic map. The set of \( a \) for which the generated sequence is periodic is dense in \([0, 2]\). Theoretically, to every couple \((a, x_0)\) corresponds one real sequence \( x_0x_1x_2... \) and one symbolic sequence \( S_0S_1S_2... \). But, it is well known [5] that sequences generated by the logistic map are very sensitive to initial conditions and to parameter variations.

![Fig. 3. Symbolic Sequence Generator](image)

For Kneading sequences, the problem related to initial condition doesn’t exit since all corresponding real sequences begin by \( x_0 = 1 \), moreover super stability of such sequences makes inaccuracy on initial condition tolerable. The theorem above tell us that even inaccuracy on parameter value doesn’t change the symbolic sequence if we consider values of \( a \) close enough to a given \( a \) corresponding to a super stable orbit.

**Theorem 1 (6)** If \( f_a(x) = 1 - a_xx^2 \) has a super-stable orbit of length \( p \), i.e \( f_a^p(0) \neq 0 \) for all \( 0 < j < p \) and \( f_a^p(0) = 0 \), then there exist a neighborhood \( U \) of \( a \), such that for all \( a \in U \), \( f_a \) has a stable periodic orbit attracting 0 with \( sign(f_a^{m+p}(0)) = sign(f_a^m(0)) \) for all \( m \geq 0 \).

We should note here that the neighborhood \( U \) can be very small and the parameter value must be very close to the real one to obtain the same symbolic sequence.

**C. Kneading Sequence Properties**

We give bellow some important properties of Kneading Sequences, proofs and details can be found in [5].

- All super stable orbits begin by the same initial condition \( x_0 = 1 \) so each Kneading sequence is defined by a parameter value \( a \).
- Infinite precision is not necessary to find \( a \) and to generate Kneading sequences.
- Complexity of generating Kneading sequences is independent of the period \( N \).

**IV. Performances of Kneading Sequences**

In this section, will be presented a performance comparison between kneading and Gold sequence of period 31.

**A. Comparison Criterion**

Based on the formulas (7-11), Figure Of Merit (FOM) criterium is considered in this work. If \( C_k(l) \) is the Aperiodic Autocorrelation Function of a given sequence \( a_k \), the FOM of \( a_k \) is

\[ FOM_k = \frac{|C_k(0)|^2}{\sum_{n=1}^{N-1} |C_k(n)|^2} \]

(14)

For a set of \( M \) sequences \( a_k \), \( k = 1, 2, ..., M \), the average FOM is defined by

\[ FOM = \frac{\sum FOM_k}{M} \]

(15)

For every sequence of both sets we considered the phase that minimizes the side lobe energy of the aperiodic autocorrelation function and the pick maximal of the odd autocorrelation function. (the phase \( k \) of the periodic sequence \( S_0S_1...S_{N-1}S_0... \) is the periodic sequence \( S_0S_1...S_{N-1}S_0... \))

In figure 4 is depicted the FOM of the two types of sequences versus the number of users. We can see the huge difference between the two curves is logical, this shows the limitation of classical sequences from correlation properties point of view. It is worth noting that we found 1692 Kneading sequences with FOM greater than 3.5. This value is achieved by only one Gold sequence.

**B. Application to Asynchronous DS-CDMA System**

As an application of the previous results the SNR given in formula (9) will be considered. We plotted in figure 5 the SNR versus \( E_b/N_0 \), for the two types of sequences and for various number of users. We can see that kneading sequences behave much better then Gold ones. Especially when \( E_b/N_0 \) is large, indeed, in this
case the influence of the interference part of the total noise becomes more dominating, thus the role of the sequence set is more important when $\frac{E_{b}}{N_0}$ is larger.

V. Conclusion

In this contribution we showed that classical sequences don’t allow desired correlation properties, especially in Asynchronous DS-CDMA application. The Figure of Merit of Gold Sequences was compared to kneading sequence one and it was found that the later is better than the former according to this criterium. The consequence of this is that the performance of asynchronous DS-CDMA system measured by the Signal to Noise Ratio is better when kneading sequences are used. Since the finite number of possible binary sequence sets is large and increase exponentially with the considered period, finding the optimum set require huge computation; this difficulty exists also in determining kneading sequences of the logistic map. The essential problem is not to find optimum sequence set, but to generate the maximum of binary sequences that have various correlation properties. This is useful when we take into account the characteristics of the channel and the type of transmission in a DS-CDMA system. Indeed, sequences should be chosen accordingly. Thus, it is important to know if it is possible to generate a given set of kneading sequence using classical Shift Register Generators, and if it is possible what is the minimum number of registers required to that.

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References

Fig. 4. SNR versus $\frac{E_b}{N_0}$ for Kneading and Gold Sequences