On the Linear Quadratic Minimum-Fuel Problem

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ABSTRACT
E.I.Verriest and F.L.Lewis have presented in [1] a new method to approach the minimum-time control of linear continous-time systems avoiding the Bang-Bang control. Their method relied on the optimization of a cost including time energy and precisions terms. Then, N.Elalami and N.Znaidi [2], extended these results to the discrete-time linear systems. The objective of this work is to propose an approach for minimal-fuel problem where the term of consumption is increased by an energetic term in order to avoid Bang-off-Bang control and singular intervals . Indeed we consider the equation of Hamilton-Jacobi-Bellman(HJB) relating to the problem of minimal consumption. And by making use of a nonlinear programming problem on a partition of $\mathbf{R}^m$, the solution of the minimum-fuel problem, according to the Riccati matrix, is reduced to the resolution of a system of differential equation.

KEY WORDS
Minimum-Fuel, Hamilton-Jacobi-Bellman, Riccati, Dc motor.

1 Introduction

In this work we consider problems in which control effort required, rather than elapsed time, is the criterion of optimality. Such problems arise frequently in aerospace applications, where often there are limited control resources available for achieving desired objectives. Let us assume that the state equations of a system are of the form

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + bu(t) \\ x(0) & = & x_0 \end{array} \right. \quad (1)$$

where A is $n\times n$ constant matrix, $x(t) \in \mathbf{R}^n$is the state vector, b is $n \times 1$ vecor, $u(t) \in \mathbf{R}$ is the control input and $t \in [0, T]$. The objective is to find the minimum control that minimizes the performance measure

$$J(u) = \int_0^T (\rho|u(t)| + ru^2(t))\, dt + x^T(T)Gx(T). \quad (2)$$

where $\rho > 0, r > 0$ and G is a constant symetric positive definite matrix.

2 Characterization of the Minimum-Fuel

In this section the objective is to find the control that minimizes (1) under (2). For this purpose, we will consider the Hamilton-Jacobi-Bellman equation associated with this problem:

$$\frac{\partial V}{\partial t} + \min_u \left\{ \rho|u| + ru^2 + \frac{\partial V}{\partial x}^T (Ax + bu) \right\} = 0. \quad (3)$$

where:

$$V(t, x(t)) = \min_{u(\tau)} \int_t^T (\rho|u(\tau)| + ru^2(\tau))\, d\tau + x^T(T)Gx(T) \quad (4)$$

is the value function.
In practice, the numerical integration of an partial derivative equation requires much calculations. We will try to give an analytical method allowing to bring back to differential equations. The optimal control problem is reduced then to the following nonlinear optimization problem:

$$\min_{u \in R} \left\{ g(u) = \rho|u| + ru^2 + \eta u + c \right\}, \quad (5)$$

therfore:

$$g(u) = \left\{ \begin{array}{ll} (\rho + \eta)u + ru^2 + c & \text{for } u \geq 0 \\ (-\rho + \eta)u + ru^2 + c & \text{for } u \leq 0. \end{array} \right.$$

From the Lagrange's theory, it is necessary that:

$$u^* = \left\{ \begin{array}{ll} 0 & \text{for } 0 \leq |\eta| \leq \rho \\ \frac{-1}{2r}(\frac{-\rho}{|\eta|} + 1)\eta & \text{for } 0 \leq \rho \leq |\eta|. \end{array} \right.$$

where u^* is the solution of (5). This yields to the following result:

Proposition 1 *The form of the optimal control is:*

$$u^* = \left\{ \begin{array}{ll} 0 & for\ 0 \leq |(\frac{\partial V}{\partial x})^T b| \leq \rho \\ \frac{-1}{2r}(\frac{-\rho}{|(\frac{\partial V}{\partial x})^T b|} + 1)(\frac{\partial V}{\partial x})^T b & for\ 0 \leq \rho \leq |(\frac{\partial V}{\partial x})^T b|. \end{array} \right.$$

which when substituted in the (HJB)(3), leads to consider the two following situations:

- *case 1:* $0 \leq |(\frac{\partial V}{\partial x})^T b| \leq \rho$

$$\frac{\partial V}{\partial t} + (\frac{\partial V}{\partial x})^T Ax = 0$$

- *case 2:* $0 \leq \rho \leq |(\frac{\partial V}{\partial x})^T b|$

$$\frac{\partial V}{\partial t} + (\frac{\partial V}{\partial x})^T Ax - \frac{1}{4r}(\rho - |(\frac{\partial V}{\partial x})^T b|)^2 = 0.$$

One way to solve the Hamilton-Jacobi-Bellman equation is to guess a form of the solution and see if it can be made to satisfy the differential equation and the boundary conditions. Therfore, it is reasonable to assume

$$V(t,x) = x^T(t)K(t)x(t) + x^T(t)s(t) + l(t) \quad (6)$$

where $K(t)$ is a real symetric positive-definite matrix, $s(t)$ is a $n \times 1$ vector and l is a scalar function, replacing the value function(4) by(6), we get:

Proposition 2 *In the case where $(\frac{\partial V}{\partial x})^T b > 0$, the solution of the minimum-fuel problem (1)-(2),according to the Riccati matrix, is reduced to the resolution of the following system:*

$$\begin{cases} \dot{K} + KA + A^T K - \frac{1}{r} K b b^T K & = 0 \\ \dot{s} + A^T s - \frac{1}{r} K b(\rho + b^T s) & = 0 \\ \dot{l} - \frac{1}{4r}(\rho - s^T b)^2 & = 0 \\ K(T) = G, \quad s(T) = 0, \quad l(T) = 0. & \end{cases} \quad (7)$$

therfore the optimal control is given in feedback form by:

$$u^* = \frac{1}{r} 2b^T K x - \frac{1}{r}(\rho - s^T b) \quad (8)$$

Moreover, the optimal equation becomes:

$$\dot{x} = (A - \frac{1}{r} b b^T K)x + \frac{1}{2r} b(\rho - b^T s)$$

and the minimal cost can be expressed by:

$$J(u^*) = x_0^T K(0) x_0 + s(0) x_0 + l(0)$$

.

Remark.1 It is known that the Riccati equation of the form

$$\dot{K} + KA + A^T K - \frac{1}{r} K b b^T K = 0 \quad (9)$$

may be transformed, in the case where K(t) is invertible $\forall t$, into a Lyapunov equation, by multiplying left and right sides of (9) by $K^{-1}(t)$.

Proposition 3 *In the case where $(\frac{\partial V}{\partial x})^T b < 0$, it is sufficient to replace ρ by $-\rho$.*

Example A set of state equations for the dc motor with constant armature current is:

$$\begin{cases} \frac{di(t)}{dt} & = & -\frac{R}{L} i(t) + \frac{1}{L} v_{in}(t) \\ \frac{dw(t)}{dt} & = & \frac{K}{I} i(t) \end{cases} \quad (10)$$

where i is the induced current, w is the angular velocity, the applied voltage v_{in} is the input to the system, R is the resistance of the circuit, L the self inductance for the armature and K, I are constants depending on certain physical properties of the motor. Defining the state variable and the control as:
$x_1(t) = i(t)$, $x_2(t) = w(t)$ and $u(t) = v_{in}(t)$
the state model of the system becomes:

$$\dot{x}(t) = \begin{pmatrix} -\frac{R}{L} & 0 \\ \frac{K}{I} & 0 \end{pmatrix} x(t) + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} u(t)$$

.

Our aim is to find the minimum control wich drives the system (10) from $x(0) = (x_{01} \quad x_{02})$ and minimizes the following cost:

$$J(u) = \int_0^T (\rho|u(t)| + ru^2(t))\, ds + Gx^2(T)0.$$

The Riccati equation, the differential vector s and the scalar function l are found from system(7) with the result:

$$\begin{cases} \dot{K_{11}} - 2\frac{R}{L} K_{11} - \frac{1}{rL^2} K_{11}^2 + 2\frac{K}{I} K_{12} & = 0 \\ \dot{K_{12}} - \frac{R}{L} K_{12} - \frac{1}{rL^2} K_{11} K_{12} + \frac{K}{I} K_{22} & = 0 \\ \dot{K_{22}} - \frac{1}{rL^2} K_{12}^2 & = 0 \\ \dot{s_1} - \frac{R}{L} s_1 + \frac{K}{I} s_2 + \frac{K_{11}}{rL}(\rho - \frac{1}{L} s_1) & = 0 \\ \dot{s_2} - \frac{K_{12}}{rL^2}(\rho - \frac{1}{L} s_2) & = 0 \\ \dot{l} - \frac{1}{4r}(\rho - \frac{1}{L} s_1)^2 & = 0 \end{cases}$$

and from system(7) the boundary conditions are:
$K(T) = G$, $s(T) = 0$ and $l(T) = 0$.
The optimal control law, obtained from (8), is

$$u^* = -\frac{1}{rL}(K_{11} x_1 + K_{12} x_2) + \frac{1}{2r}(\rho - \frac{1}{L} s_1)$$

finally, noting that with the control u^* the states are given by:

$$\begin{cases} \dot{x_1} = -(\frac{R}{L} - \frac{1}{rL^2} K_{11}) x_1 - \frac{1}{rL^2} K_{12} x_2 + \frac{1}{2rL}(\rho - \frac{1}{L} s_1) \\ \dot{x_2} = -\frac{K}{I} x_1 \end{cases}$$

3 Fuel-optimal problem with multiple-input

In this section, we placed within a more general framework, in which the process to be controlled is described by the state equation:

$$\left\{\begin{array}{lcl} \dot{x}(t) & = & Ax(t)+Bu(t) \\ x(0) & = & x_0 \end{array}\right. \tag{11}$$

where A and B are $n\times n$ and $n\times m$ matrices and the performance measure to be minimized is:

$$\min_u J(u)=\int_0^T(\rho\sum_{i=1}^m|u_i|+u^TRu)(t)dt+x^T(T)Gx(T). \tag{12}$$

For a system (11) with several input, such a (HJB) equation would have the following form

$$\frac{\partial V}{\partial t}+\min_u\left\{\rho\sum_{i=1}^m|u_i|+u^TRu+(\frac{\partial V}{\partial x})^T(Ax+Bu)\right\}=0 \tag{13}$$

this leads to solve the following nonlinear programming problem:

$$\min_{u\in\mathbf{R}^m}\left\{g(u)=\rho\sum_{i=1}^m|u_i|+u^TRu+(\frac{\partial V}{\partial x})^T(Ax+Bu)\right\}. \tag{14}$$

In view to use the previous techniques, the idea is to solve the programming problem on a partition of the set $\mathbf{R}^m$:

$$\min_{u\in\mathbf{R}^m}g(u)=Inf(\min_{u\in\mathcal{P}}g(u))_{\mathcal{P}\subset\mathbf{R}^m},\cup\mathcal{P}=\mathbf{R}^m \tag{15}$$

this suggests to introduce these 2^m applications:

$$\{\sigma_k\}_{k=1}^{2^m}$$

with: $\sigma_k(i)\in\{1,-1\}\quad\forall i\in I\quad\forall k\in L=\{1,\cdots,2^m\}$ and gives 2^m parts of $\mathbf{R}^m$:

$$\{\mathcal{P}_k\}_k^{2^m}$$

each σ_k corresponds to the $\mathcal{P}_k$, indeed:

$$u=(u_1,\cdots,u_m)\in\mathcal{P}_k\Leftrightarrow\sum_{i=1}^m|u_i|=\sum_{i=1}^m\sigma_k(i)u_i.$$

We now define $\bar{\rho_k}$ and $\mathcal{M}_{\sigma_k}$ as follows:

$$\bar{\rho_k}=\rho\begin{pmatrix}\sigma_k(1)\\ \vdots\\ \sigma_k(m)\end{pmatrix}\in\mathbf{R}^m,$$

$$\mathcal{M}_{\sigma_k}=\begin{pmatrix}\sigma_k(1) & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & \sigma_k(m),\end{pmatrix}$$

Thus system (15) becomes:

$$\min_{u\in\mathbf{R}^m}\left\{\begin{array}{l} u^T\bar{\rho_k}+u^TRu+(\frac{\partial V}{\partial x})^T(Ax+Bu)\\ \varphi_k(u)=-\mathcal{M}_{\sigma_k}u\le 0\end{array}\right\}_{k=1}^{2^m}.$$

To use the theory of nonlinear programming, we first form the Lagrangien:

$$L(u,\lambda)=g(u)+\lambda^T\varphi_k(u)=g(u)+\sum_{i=1}^m\lambda_i\varphi_{ik}(u)$$

then we obtain a linear programming problem:

$$\left\{\begin{array}{l}\bar{\rho_k}+2Ru^*+B^T\frac{\partial V}{\partial x}-\mathcal{M}_{\sigma_k}\lambda=0\\ \lambda_i=0\text{ou}\quad u_i=0\quad\forall i=1,\cdots,m\end{array}\right. \tag{16}$$

Proposition 4 *The form of the optimal control is:*

$$u^*=-\frac{1}{2}R^{-1}\mathcal{N}_{\sigma_k}$$

which when substituted in the Hamilton-Jacobi-Bellman(13) gives:

$$\frac{\partial V}{\partial t}-(\frac{\partial V}{\partial x})^TAx-\frac{1}{4}\mathcal{N}_{\sigma_k}^TR^{-1}\mathcal{N}_{\sigma_k}=0 \tag{17}$$

where:

$$\bar{\rho_k}=\mathcal{M}_{\sigma_k}\bar{\rho_1},\ \bar{\rho_1}=\rho\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}.$$

and $\mathcal{N}_{\sigma_k}=\mathcal{M}_{\sigma_k}(\bar{\rho_1}-\lambda)+B^T\frac{\partial V}{\partial x}$
Let us assume that a solution is of the form:

$$V(t,x)=x^T(t)K(t)x(t)+x^T(t)s(t)+l(t)$$

then we have:

Proposition 5 *the solution of the minimum-fuel problem (11)-(12),according to the Riccati matrix, is reduced to the resolution of the following system:*

$$\left\{\begin{array}{lcl}\dot{K}+KA+A^TK-KBR^{-1}B^TK & = & 0\\ \dot{s}+A^Ts-KBR^{-1}\mathcal{H}_{\sigma_k} & = & 0\\ \dot{l}-\frac{1}{4}\mathcal{H}_{\sigma_k}^TR^{-1}\mathcal{H}_{\sigma_k} & = & 0\\ K(T)=G,\quad s(T)=0,\quad l(T)=0. & & \end{array}\right.$$

therfore the optimal control is given in feedback form by:

$$u^*=-R^{-1}B^TKx-\frac{1}{2}R^{-1}\mathcal{H}_{\sigma_k}$$

where:
$\mathcal{H}_{\sigma_k}=\mathcal{M}_{\sigma_k}(\bar{\rho_1}-\lambda)+B^Ts$

4 Application to the minimum-fuel problem of two input

In this section we are interested in the case $m=2$. Thus the linear programming problem(16) yields to the following:

$$\left\{\begin{array}{l}\bar{\rho_k}+2Ru^*+B^T\frac{\partial V}{\partial x}-\mathcal{M}_{\sigma_k}\lambda=0\\ \lambda_i=0\quad\text{ou}\quad u_i=0\quad\forall i=1,2.\end{array}\right.$$

We find several cases for each of these following parts:

$$\mathcal{P}_1 =]-\infty,0]\times]-\infty,0], \quad \mathcal{P}_2 =]-\infty,0]\times[0,+\infty[$$
$$\mathcal{P}_3 = [0,+\infty[\times]-\infty,0], \quad \mathcal{P}_4 = [0,+\infty[\times[0,+\infty[.$$

Assume that the symmetric and positive definite matrix R is of the form:

$$R = \begin{pmatrix} r_{11} & 0 \\ 0 & r_{22} \end{pmatrix}$$

we consider the space $\mathbf{R}^2$ endowed with the norm:

$$|x| = |x_1| + |x_2|$$

where: $x = (x_1 \quad x_2)$
in this section we investigate also the use of the Hamilton-Jacobi-Bellman equation and the nonlinear programming as a means of solving the minimum-fuel problem of two-dimentionsional control, Then we have:

Proposition 6 *The form of the optimal control is:*

$$u^* = \begin{cases} 0 & \text{for } |\eta| \le \rho \\ \frac{1}{2}R^{-1}(\rho-\eta) & \text{for } \rho \le |\eta| \text{ and } \eta \ge 0 \\ \frac{-1}{2}R^{-1}(\rho+\eta) & \text{for } \rho \le |\eta| \text{ and } \eta \le 0 \\ \frac{-1}{2}R^{-1}\begin{pmatrix} -\rho+\eta_1 \\ \rho+\eta_2 \end{pmatrix} & \text{for } \rho \le |\eta| \text{ , } \eta_2 \le 0 \le \eta_1 \\ \frac{-1}{2}R^{-1}\begin{pmatrix} \rho+\eta_1 \\ -\rho+\eta_2 \end{pmatrix} & \text{for } \rho \le |\eta| \text{ , } \eta_1 \le 0 \le \eta_2 \\ \frac{1}{2r_{22}}\begin{pmatrix} 0 \\ \rho-\eta_2 \end{pmatrix} & \text{for } |\eta_1| \le \rho \le |\eta_2| \text{ , } \eta_2 \ge 0 \\ \frac{-1}{2r_{22}}\begin{pmatrix} 0 \\ \rho+\eta_2 \end{pmatrix} & \text{for } |\eta_1| \le \rho \le |\eta_2| \text{ , } \eta_2 \le 0 \\ \frac{-1}{2r_{11}}\begin{pmatrix} \rho-\eta_1 \\ 0 \end{pmatrix} & \text{for } |\eta_2| \le \rho \le |\eta_1| \text{ , } \eta_1 \ge 0 \\ \frac{-1}{2r_{11}}\begin{pmatrix} \rho+\eta_1 \\ 0 \end{pmatrix} & \text{for } |\eta_2| \le \rho \le |\eta_1| \text{ , } \eta_1 \le 0 \end{cases}$$

which when substituted in the (HJB)(17), leads to consider the two following situations:

- *case 1:* $\quad 0 \le |(\frac{\partial V}{\partial x})^T b| \le \rho$

$$\frac{\partial V}{\partial t} + (\frac{\partial V}{\partial x})^T Ax = 0$$

- *case 2:* $\quad \rho \le |\eta| \qquad \eta \le 0$

$$\frac{\partial V}{\partial t} + (\frac{\partial V}{\partial x})^T Ax - \frac{1}{4}\mathcal{N}_{\sigma_1}^T R^{-1}\mathcal{N}_{\sigma_1} = 0$$

$$\eta = B^T(\frac{\partial V}{\partial x}), \quad c = (\frac{\partial V}{\partial x})^T Ax.$$

In addition we have the optimal control in feedback form, by making use of the Riccati equation.

Proposition 7 *Then optimal fuel consomed is:*

$$u^* = -R^{-1}B^T Kx - \frac{1}{2}\mathcal{H}_{\sigma_1}$$

where:

$$\begin{cases} \dot{K} + KA + A^T K - KBR^{-1}B^T K & = 0 \\ \dot{s} + A^T s - KBR^{-1}\mathcal{H}_{\sigma_1} & = 0 \\ \dot{l} - \frac{1}{4}\mathcal{H}_{\sigma_1}^T R^{-1}\mathcal{H}_{\sigma_1} & = 0 \\ K(T) = G, \quad w(T) = 0, \quad l(T) = 0. \end{cases}$$

finally, the state equation become

$$\dot{x} = (A - BR^{-1}B^T K)x - \frac{1}{2}BR^{-1}\mathcal{H}_{\sigma_1}$$

with:
$\mathcal{N}_{\sigma_1} = \bar{\rho_1} + B^T\frac{\partial V}{\partial x}$ and $\mathcal{H}_{\sigma_1} = \bar{\rho_1} + B^T s$
Remark.2 For the other remaining situations, we find the case ($\rho \le |\eta|$ and ($\mathcal{M}\sigma_1\eta \ge 0$ or $\mathcal{M}\sigma_3\eta \ge 0$ or $\mathcal{M}\sigma_4\eta \ge 0$)) for wich it is sufficient to replace respectively $\bar{\rho_1}$ by ($-\bar{\rho_1}$ or $\mathcal{M}\sigma_4\bar{\rho_1}$ or $\mathcal{M}\sigma_3\bar{\rho_1}$).
And the case $|\eta_1| \le \rho \le |\eta_2$for wich it is enough to replace R^{-1} by $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}R^-1$ respectively by $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}R^-1$ for the case $|\eta_2| \le \rho \le |\eta_1|$ moreover ρ can take the value $-\rho$.

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