Application of Perfect Distribution Phenomenon for Acoustics and Music

V.Riznyk
Department of Telecommunications and Electrical Engineering
University of Technology and Agriculture in Bydgoszcz
7 Al. Kaliski, 85-796 Bydgoszcz
POLAND

Abstract: - The paper presents new mathematical principle, based on the Perfect Distribution Phenomenon, namely the concept of Perfect Ring "Vyazankas" (PRV)s, which can be used for finding of the optimal placement of structural elements in spatially or temporally distributed acoustic or underwater acoustic systems and generalization of these methods and results to the improvement and optimization of the systems, including positioning of elements in the system (e.g. an active sonar) with respect to resolving ability, and the other operating characteristics of the system, as well as high-performance coding in acoustics and music.

Key words: – Perfect Distribution Phenomenon, Perfect Ring "Vyazankas", structural optimization, combinatorial construction, resolving ability, Golomb ruler, Perfect 3-D "Vyazanka", 3-D Non-redundant structure, Perfect 3-D Monolithic Code.

1 Introduction

Problem of structural optimization of an acoustic systems relates to finding the best placement structural elements in spatially or temporally distributed systems. The problem to be of great important for improving the quality indices of the system, including active sonar, and it is closely connected with application of fundamental research in combinatorial theory [1]. Research into underlying mathematical area involves investigation of novel techniques based on combinatorial mathematics [2] and the Perfect Distribution Phenomenon, namely the concept of the Perfect Numerical "Vyazankas" (PNV) [3], which can be used for finding optimal solutions for some problems in acoustics.

It is known, finite-field theory and appropriate technique, based on wide-range sonar interferometric synthesis using non-redundant masks [4]. Some regular methods for constructing non-redundant two-dimensional *n*-elements masks over $n \times n$ grids, based on special combinatorial structures known as difference sets [1] are suggested in the publication [4]. Application of modern combinatorial analysis obtains a lot of problems of signal processing for radar, sonar and data communications [2]. However, the classical theory of combinatorial configurations such as difference sets, finite projective geometry and general block design theory can hardly be expected effective for solving 3-D problem using methods, based on the theory. Hence, both an advanced theory and regular method for finding optimal solution the problem are needed.

2 Problem Formulation

Research into the underlying combinatorial principle relates to development both the advanced combinatorial 3-D theory and regular method for design of synthetic-aperture sonar with high-resolution sounder, based on application of remarkable combinatorial properties of Perfect Numerical "Vyazankas", and generalization of these methods and results to the improvement of quality indices of acoustic or sonar systems (e.g. active sonar) with respect to resolving ability and operating range.

1-D Perfect Numerical Vyazanka (1-D PNV) is an ordered numerical construction with n distinct integers, which form perfect partitions of finite interval [1,s] of integers. The sums of connected sub-sequences of 1-D PNV enumerate the set integers [1,s] exactly R-times.

Example: The Golomb Ideal Ruler with marks 0,1,4,6 containing three intersections

{1,3,2}, where 1=1-0, 3=4-1, and 2=6-4, allows an enumeration of all numbers 1=1, 2=2, 3=3, 4=1+3, 5=3+2, 6=1+3+2 exactly once (R=1).

Here is a non-redundant 1-D mask, based on the Golomb Ideal Ruler with marks 0,1,4,6:

Marks:	0	1	2	3	4	5	6
	◎	◎			◎		◎

Fig.1. A non-redundant 1-D mask, based on the Golomb Ideal Ruler with marks 0,1,4,6.

We say the numerical chain sequence {1,3,2} is 1-D PNV of chain-like topology.

Unfortunately, there are not exist Golomb Ideal Rulers with more of three intersections. The problem, is known, to be of great important in development of regular method based on the idea of “perfect vyazanka” for finding optimal placement of structural elements in 3-D spatially distributed systems (e.g. an acoustic system or sonar) with respect to improving the resolving ability and tuning range.

3 Problem Solution

Let us consider the numerical n-stage chain sequence of distinct positive integers $\{k_1, k_2,\ldots, k_n\}$, where we require all terms in each sum to be consecutive elements of the sequence. We call this a chain numerical “vyazanka”. Easy to see the maximum number C of such sums is equal to the sum $k_1 + k_2 +\ldots+ k_n = C$ of all integers. If we regard the chain sequence as being cyclic, so that k_n is followed by k_1, we configure a ring-like vyazanka. A sum of consecutive terms in the ring-like vyazanka can have any of the n terms as its starting point, and can be of any number of terms from 1 to n-1. In addition, there is the sum of all n terms, which is the same independent of the starting point. So, the maximum number of distinct sums S of consecutive terms of the ring-like vyazanka is given by

$$S = n(n-1) + 1 \qquad (1)$$

Comparing the maximum number $C= n(n+1)/2$ of distinct sums of the chain-like and ring-like numerical vyazanka, we can see that the number of sums S for consecutive terms in the ring topology is nearly double the number of sums C in the daisy-chain topology, for the same vyazanka of n terms. Hence vyazanka with the ring topology provides an ability to reproduce more of combinatorial varieties in the numerical constructions with a fixed number of elements and limited number of bonds.

An n-stage sequence $\{k_1, k_2,..., k_n\}$ of natural numbers for which the set of all S circular sums consists of the numbers from 1 to $S = n(n-1)+1$ (each number occurs exactly once) is called a simple (R=1) "Perfect Ring Vyazanka" (PRV).

Here is an example of a simple PRV with $n = 4$ and $S = 13$, namely {1,4,6,2,}:

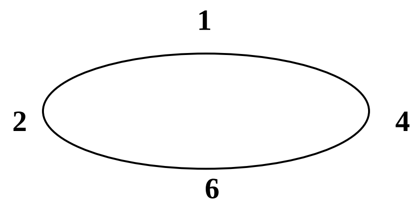

Fig.2. Simple “Perfect Ring Vyazanka” {1,4,6,2}

To see this, we observe:

Table 1.

Table of circular sums for PRV {1,4,6,2}

1 = 1	2 = 2	3 = 2+1
4 = 4	5 = 1+4	6 = 6
7 = 2+1+4	8 = 6+2	9 = 6+2+1
10 = 4+6	11= 1+4+6	12 = 4+6+2

13=1+4+6+2

We see that Tabl.1 contains each value as a circular sum from 1 to 12 exactly once (R=1).
Note that if we allow summing over more than one complete revolution around the ring, we can all positive integers as such sums.
Thus:
14 = 1 + 4 + 6+ 2 +1, 15 = 2 + 1 + 4 + 6 + 2, etc.

Next, we consider a more general type of PRV, where the S ring-sums of consecutive terms give us each integer value from 1 to N, for some integer N, exactly R times, as well as the value N+1 (the sum of all n terms) exactly once. Here we see that:

$$N = n(n-1)/R \qquad (2)$$

An example with $n = 4$ and $R = 2$, so that $N = 6$, is the ring sequence {1,1,2,3}, for which the sums of consecutive terms are as follows:

Table 2.
Table of circular sums for PRV {1,1,2,3}

1 = 1 1 = 1	2 = 2 2 = 1 + 1	3 = 3 3 = 1 + 2
4 = 3 + 1 4 = 1 + 1 + 2	5 = 2 + 3 5 = 3 + 1 + 1	6 = 1 + 2 + 3 6 = 2 + 3 + 1

7 = 1+1 + 2 + 3

We see that each "circular sum" from 1 to 6 occurs exactly twice (R = 2). We say that this PRV has the parameters n = 4, R = 2.

Here is an example of the simple two-dimensional PRV {(1,1), (1,2), (1,4), (1,3)}. Using circular two-dimensional vector-sums, it is easy to calculate all the sums, taking modulus m_1 = 4 for the first component of the vector-sums and modulus m_2 = 5 for the second component of the vector-sums:

Table 3.
Table of circular vector- sums for 2-D PRV {(1,1), (1,2), (1,4), (1,3)}

(1,1)= (1,1)	(2,1)= (1,2)+(1,4)	(3,1)= (1,3)+(1,1)+(1,2)
(1,2)= (1,2)	(2,2)= (1,4)+(1,3)	(3,2)= (1,1)+(1,2)+(1,4)
(1,3)= (1,3)	(2,3)= (1,1)+(1,2)	(3,3)= (1,4)+(1,3)+(1,1)
(1,4)= (1,4)	(2,4)= (1,3)+(1,1)	(3,4)= (1,2)+(1,4)+(1,3)

So long as the elements of the Perfect Ring Vyazanka themselves are circular vector-sums too, the result of the calculation forms 4 × 3 – matrix, which exhausts the circular 2-D vector-sums and each of its meets exactly once (R=1).

Table 4.
4 × 3 – matrix of circular vector- sums for 2-D PRV {(1,1), (1,2), (1,4), (1,3)}

(1,1)	(2,1)	(3,1)
(1,2)	(2,2)	(3,2)
(1,3)	(2,3)	(3,3)
(1,4)	(2,4)	(3,4)

Hence, the ring sequence of the vectors {(1,1), (1,2), (1,4), (1,3)} is 2-D PRV with n = 4, R = 1, m_1 = 4, m_2 = 5.

3.1 3-D Non-redundant Structures

3-D (t=3) and multidimensional PRVs are ring-like sequences of t-stage (t-D) ordered sub-sequences (integer vectors), which form "perfect" t-D partitions of a finite t-D space interval of the vectors from (0,…0) to (N_1,…N_t) exactly R-times. So, any n-stage ring sequence {K_1, K_2,…K_i,…K_n} of terms K_i = {k_{i1},k_{i2},k_{i3}}, for which set of all circular 3-D sums of the consecutive terms (sums is calculated, its own modulo m_j, j=1,2,3) enumerate a set of 3-stage terms, configure 3-D grid, each node of the grid meets exactly R-times.

Example: The ring-ordered sequence of 3-stage terms {(1,1,1), (1,1,2), (1,0,3), (0,2,2), (0,1,4), (0,2,4)} forms 2 × 3 × 5 – matrix, which exhausts the circular 3-D vector-sums, taking m_1 = 2, m_2 = 3, m_3 = 5, and each of its meets exactly once (R=1):

{0,0,0}= (1,1,2)+(1,0,3)+(0,2,2)+(0,1,4)+(0,2,4);
{0,0,1}= (0,2,2)+(0,1,4);
{0,0,2}= (1,1,2)+(1,0,3)+(0,2,2);
{0,0,3}= (0,1,4)+(0,2,4);
{0,0,4}= (1,0,3)+(0,2,2)+(0,1,4)+(0,2,4)+(1,1,1);
{0,1,0}= (1,1,2)+(1,0,3);
{0,1,1}= (1,1,2)+(1,0,3)+(0,2,2)+(0,1,4);
{0,1,2}= (1,1,2);
{0,1,3}= (0,2,2)+(0,1,4)+(0,2,4)+(1,1,1)+(1,1,2);
{0,1,4}= (0,1,4);
{0,2,0}= (0,2,2)+(0,1,4)+(0,2,4);
{0,2,1}= (0,1,4)+(0,2,4)+(1,1,1)+(1,1,2);
{0,2,2}= (0,2,2);
{0,2,3}= (1,1,1)+(1,1,2);
{0,2,4}= (0,2,4);
{1,0,0}= (0,2,4)+(1,1,1);
{1,0,1}= (0,2,2)+(0,1,4)+(0,2,4)+(1,1,1);
{1,0,2}= (0,2,4)+(1,1,1)+(1,1,2)+(1,0,3)+(0,2,2);
{1,0,3}= (1,0,3);
{1,0,4}= (1,0,3)+(0,2,2)+(0,1,4);
{1,1,0}= (0,2,4)+(1,1,1)+(1,1,2)+(1,0,3);
{1,1,1}= (1,1,1);
{1,1,2}= (1,1,2);
{1,1,3}= (1,1,1)+(1,1,2)+(1,0,3)+(0,2,2);
{1,1,4}= {1,1,4};
{1,2,0}= (1,0,3)+(0,2,2);
{1,2,1}= (1,1,1)+(1,1,2)+(1,0,3);
{1,2,2}= (1,1,1)+(1,1,2)+(1,0,3)+(0,2,2)+(0,1,4);
{1,2,3}= (1,0,3)+(0,2,2)+(0,1,4)+(0,2,4);
{1,2,4}= (0,1,4)+(0,2,4)+(1,1,1)+(1,1,2)+(1,0,3).

Let the first of six (n = 6) mask elements is the (0,0,0) cell of 2 × 3 × 5 – matrix cycling. Now, we can obtain coordinates of the remaining five elements accordingly the underlying 3-D perfect distribution cycling, modulus m_1 = 2, m_2 = 3, m_3=5:

(1,1,1), (0,2,3), (1,2,1), (1,1,3), (1,2,2).

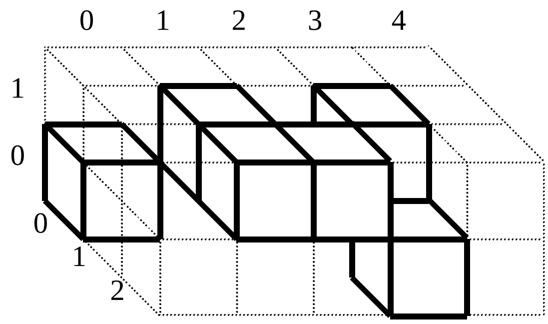

Fig.3. A non-redundant 2 × 3 × 5 – matrix cycling, based on the PRV {(1,1,1), (1,1,2), (1,0,3), (0,2,2), (0,1,4), (0,2,4)}.

Now, to obtain configuration with smaller grids, we can exclude all right-hand columns (Fig.3), and one can be reconstructed on smaller matrix 2 × 3 × 4.

It is exists an infinite set of the PRV and values of its parameters can be of any large number. Underlying technique can be used both for design acoustic or sonar systems with high quality indices due to all spacing vectors between their elements are different in order to avoid of interference of components of the same spatial frequency, and for development methods of non-redundant 3-D mask construction.

3.2 Perfect 3-D Monolithic Code

Underlying combinatorial construction can be represented as mathematical model of optimum coding system, based on so-called "Monolithic Binary Code" (MBC). This code forms binary code combinations which all symbols "1" as well as symbols "0" are arranged together [5]. The PRVs provide an optimal model of the coding system.

Here is an example of Perfect 3-D MBC coding system based on the table of circular sums to be of PRV {(1,1,1), (1,1,2), (1,0,3), (0,2,2), (0,1,4), (0,2,4)} on the 2 × 3 × 5 – matrix cycling.

The ring-like 3-D MBC formed on the PRV {(1,1,1), (1,1,2), (1,0,3), (0,2,2), (0,1,4), (0,2,4)} is illustrated by the next table.

Table 5.

3-D MBC coding system based on PRV {(1,1,1), (1,1,2), (1,0,3), (0,2,2), (0,1,4), (0,2,4)}

(000)= 011111	(020)= 000111	(110)= 111001
(001)= 000111	(021)= 110011	(111)= 100000
(002)= 011100	(022)= 000100	(112)= 010000
(003)= 000011	(023)= 110000	(113)= 111100
(004)= 101111	(024)= 000001	(114)= 000010
(010)= 011000	(100)= 100001	(120)= 001100
(011)= 011110	(101)= 100111	(121)= 111000
(012)= 010000	(102)= 111101	(122)= 111110
(013)= 110111	(103)= 001000	(123)= 001111
(014)= 000010	(104)= 001110	(124)= 111011

Tabl.6 contains the set of binary code combinations for coding of all 3-D vectors on the integer 2 × 3 × 4 - matrix from (0,0,0) = (code 011111) to (1,2,4) = (code 111011), where each of them has been coded in circular 3-D MBC. The code allows coding of 3-D acoustic signals, using the smallest possible number of connected symbols “1” in binary combinations as well as development of the mathematical models in music, involving difference set theory [6]. The remarkable property of the 3-D MBC provides its some advantages over the rest codes. One of them is simplicity of error detecting and correcting [5].

4 Conclusion

The *Perfect Numerical “Vyazankas”* (PNV) provide, essentially, a new conceptual model of 3-D acoustic systems for its investigation and development, based on the Perfect Distribution Phenomenon and remarkable properties of multidimensional PNVs. The favorable qualities of the combinatorial structures make it possible to configure novel high-performance acoustic systems using research into the underlying mathematical principles relating to the optimal placement of structural elements in spatially or temporally distributed systems.

.

References:

[1] Hall, M.Jr., *Combinatorial Theory*, Waltham, Mass.: Blaisdel Publishing Company, 1967.

[2] Golomb, S.W., Applications of Combinatorial Mathematics to Communication Signal Design, *Proceedings of the IAM Conference on Applications of Combinatorial Mathematics*, London, U.K., 1995.

[3] Riznyk, V.V., Multi-dimensional Systems Based on Perfect Combinatorial Models, *IEE, Multidimensional Systems*: *Problems and Solutions*, 1998, pp. 5/1-5/4.

[4] Kopilovich L.E., Construction of Nonredundant Masks over Square Grids Using Difference Sets, *Optics Communications*, Vol.68, No.1, 1988, pp.7-10.

[5] Golomb S.W., Osmera P., Riznyk V., Combinatorial Sequencing Theory for Optimization of Signal Data Vectors Converting and Signal Processing, *Proceedings of Workshop on Design Methodologies for Signal Processing*, Zakopane, Poland, 1996, pp.43-44.

[6] Rogers J., Mitchell B., A problem in mathematics and music, *Amer.Math.Monthly*, Vol.75, No.8, 1968, pp. 871-873.