Choice of Basic State in the Semi-Implicit Schemes

BOURCHTEIN ANDREI, KADYCHNIKOV VLADIMIR
Mathematical Department, Meteorological Department
Pelotas State University
Campus Universitario da UFPel, 96010-900 Capao do Leao-RS
BRAZIL

Abstract: - The linear stability of the semi-implicit primitive equation models with a central finite-difference vertical scheme is investigated. The absolute instability may arise by reason of a difference between the real temperature vertical profile and the basic profile, which is necessary for construction of the semi-implicit method. It is shown that the greater static stability of the basic profile in comparison with the real one is necessary condition for numerical stability at any time step \( \tau \) in order to any vertical resolution of an atmospheric model may be used. This condition is necessary and sufficient for stability when \( \tau \to \infty \).

Key-Words: - Hydrostatic atmospheric model, semi-implicit numerical scheme, numerical stability

1 Introduction

The use of explicit schemes for numerical integration of the hydrothermodynamic equations with the aim of simulation of atmospheric processes and weather forecast imposes very rigid restrictions on a time step of the schemes. For example, into the Courant stability condition on the staggered grid “C”

\[
\sqrt{2}(U + 2c) \frac{\tau}{h} \leq 1
\]

cogether with the scheme parameters \( \tau \) (time step) and \( h \) (horizontal space step) the wind speed \( U \) and the perturbations propagation speed in the linearized hydrodynamic atmospheric model \( c \) enter [9,11]. This is a speed of the most rapid waves in a hydrostatic atmosphere - the horizontally propagating Lamb waves and it is equal to the speed of sound [6]. In the Courant condition the number \( \sqrt{2} \) enters because the problem is two-dimensional in horizontal and number 2 because the grid “C” is used. When \( c = 330ms^{-1}, U = 60ms^{-1}, h = 100km \) the time step is about 1.5 min which not at all corresponds to the time scale of large-scale atmospheric processes.

Therefore, the schemes that approximate the adjustment operator implicitly are preferential [10,12]. This approach, on one hand, turns out to be relatively simple in realization because it is a linear operator which is approximated implicitly, and on the other, the parameter \( c \) is eliminated from the Courant criterion because it enters there just in virtue of the explicit approximation of linear operator. Of course, the realization of every time step becomes more complicated now. In order to calculate the values of meteorological parameters, one has to resolve a system of linear algebraic equations of high order because there are not separate formulae in every node of grid. But the modern methods of resolution of such systems [4,5,15] make these schemes much more economic than explicit schemes, anyway.

At the same time, the nonlinear advection operator can be approximated explicitly. This imposes a limitation on the parameters of a numerical scheme that is not too rigid. With the same values of \( U \) and \( h \), the value of \( \tau \) becomes about 20 min. The use of the semi-Lagrangian methods permits to eliminate this restriction, too [1,13].

As far as the coefficients of the adjustment operator in reality depend on the horizontal coordinates and time, only one his part which corresponds to a basic state of the atmosphere can be approximated implicitly. All terms describing deviations from the basic state, as well as, perhaps nonlinear terms describing the advection, are approximated explicitly. Such schemes are designated semi-implicit.

It is well known, that in the case without advection, the semi-implicit scheme is absolutely stable if the basic and real states coincide. But as pointed out by Burridge [2] computational instability may arise by reason of unavoidable differences between the real state, that is a function of the horizontal coordinates and time, and the basic state, that is constructed for a model one time for ever and therefore is fixed. In paper of Simmons et al. [14] has been found that the semi-implicit instability is
absolute, i.e., it does not disappear when \( \tau \to 0 \). But it can be avoided if an isothermal state with greater temperature is choice as a basic one. Côté et al. [3] have considered the matter analytically for a finite-element method. For the simple case without advection (\( U = 0 \)) nor the Coriolis parameter (\( f = 0 \)), which was just studied in [14], the sufficient condition on the two vertical temperature profiles to prevent semi-implicit instability in the limit of \( \tau \to 0 \) has been obtained in the following simple form: the static stability of the basic state must be greater than that of the explicit one.

The present article considers the matter for a simple central difference vertical scheme that has a widespread acceptance amongst modelers. It is shown that the simple condition obtained in [3] is necessary to prevent instability for any value of \( \tau \) and it is necessary and sufficient when \( \tau \to \infty \).

In Section 2 the finite difference formulation of the semi-implicit method for the case with the different basic and real temperature profiles is outlined. In Section 3 an algebraic equation for the amplification factor of the problem is deduced. It is shown that if both profiles coincide then all factors are situated on the unit circumference, i.e., the semi-implicit method is absolutely stable. In Section 4 the implicit method is absolutely stable. In Section 5 we show that this condition is necessary and sufficient condition on the two vertical temperature profiles to prevent semi-implicit instability in the limit case \( \tau \to \infty \). A necessary and sufficient condition on the temperature profiles of two states which guarantees computational stability for any vertical model discretization is deduced. Under natural constraints on other parameters of the profiles, the obtained condition consists in the greater static stability of the basic temperature profile in comparison with the real one. In Section 5 we show that this condition is necessary for stability when any time step is used.

### 2 Difference equations of the problem

We take the initial equations in the isobaric coordinates system omitting the advection terms and the Coriolis force. Under these conditions the momentum, hydrostatic, continuity and thermodynamic equations have the form [6]

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\partial \Phi}{\partial x} &= 0, \\
\frac{\partial v}{\partial t} + \frac{\partial \Phi}{\partial y} &= 0, \\
\frac{\partial \Phi}{\partial t} + \frac{\partial T}{\partial t} \frac{\partial \ln \zeta}{\partial t} &= -RT, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial z} &= 0, \\
\frac{\partial T}{\partial t} - S \omega &= 0.
\end{align*}
\]

The superior and inferior boundary conditions are correspondingly

\[
\omega \big|_{z=0} = 0; \quad \left( \frac{\partial \Phi}{\partial t} - c^2 \omega \right) \bigg|_{z=-1} = 0.
\]

In (1) and (2) \( \zeta = p/p_0, \rho_0 = 1000mb \). \( \Phi \) is the geopotential of surfaces \( \zeta = \text{const} \), \( T \) the temperature, \( \omega = \partial \zeta / \partial t \) the vertical velocity. The parameters \( S \) and \( c^2 \) are

\[
S = \frac{RT(1 - \Gamma)}{g \zeta}, \quad c^2 = RT|_{z=-1}
\]

where \( R \) is the gas constant of air, \( g \) the gravity acceleration, \( \Gamma \) the vertical lapse rate, \( \Gamma_0 \) the adiabatic lapse rate.

From the first two equations we construct the divergence equation

\[
\frac{\partial D}{\partial t} + \nabla \cdot \Phi = 0 \quad (D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}),
\]

which has not vertical derivatives and, therefore, will not depend from vertical finite-difference approximation.

Now, eliminating \( T \) from the third and fifth equations, we obtain

\[
\frac{\partial}{\partial \ln \zeta} \left( \frac{\partial \Phi}{\partial t} \right) = RS \omega
\]

The forth equation we rewrite as

\[
\frac{\partial}{\partial \zeta} \omega = -D.
\]

We shall approximate the vertical derivatives in the equations (4) and (5) on a staggered grid. The functions \( u, v, \omega, \Phi \) are determinate at odd levels \( \zeta_{2n-1} \) and the function \( T \) is determinate at even levels \( \zeta_{2n} \). The odd levels are chosen arbitrarily \( (\zeta_1 < \zeta_2 < \ldots < \zeta_{2N-1} = 1) \) and the even levels are calculated by formulas \( \zeta_{2n} = (\zeta_{2n-1} + \zeta_{2n+1})/2 \), \( n = 2, 4, \ldots, 2N - 2 \). We write the both equations on the temperature levels as follows

\[
\left( \frac{\partial \Phi}{\partial t} \right)_{2n+1} - \left( \frac{\partial \Phi}{\partial t} \right)_{2n-1} = RS_2 \omega_{2n+1} + \omega_{2n-1}, \quad 1 \leq n \leq N - 1;
\]

\[
\omega_{2n+1} - \omega_{2n-1} = -\frac{D_{2n+1} + D_{2n-1}}{2}, \quad 1 \leq n \leq N - 1.
\]

Resolving the system (6) and inferior boundary condition (2) with respect to \( \partial \Phi / \partial t \) we have

\[
\frac{\partial \Phi}{\partial t} - A \omega = 0.
\]

In this equation \( N \)-dimensional vectors \( \Phi \) and \( \omega \) with elements \( \Phi_{2n-1}, \omega_{2n-1} \) are introduced; \( A \) is the superior three-cornered \( N \times N \) matrix.
The equation (3), written in the geopotential levels, the matrices the real profile values by index real temperature profile enters. Let us characterize in the definition of the parameters the basic temperature profile should be introduced implicit method, beyond the real temperature profile, (2) with respect to \( \omega \) we have
\[
\mathbf{w} + \mathbf{BD} = 0. \tag{9}
\]
In this equation \( N \)-dimensional vector \( \mathbf{D} \) with elements \( D_{2n-1} \) is introduced; \( \mathbf{B} \) is the inferior three-cornered \( N \times N \) matrix
\[
\mathbf{B} = \begin{pmatrix}
 b_1 & 0 & 0 & \cdots & 0 & 0 \\
b_1 + b_2 & b_2 & 0 & \cdots & 0 & 0 \\
b_1 + b_2 & b_2 + b_3 & b_3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
b_1 + b_2 & b_2 + b_3 & b_3 + b_4 & \cdots & b_{N-1} & 0 \\
b_1 + b_2 & b_2 + b_3 & b_3 + b_4 & \cdots & b_{N-1} + b_N & b_N
\end{pmatrix}
\]
where
\[
b_1 = \zeta_1; \quad b_n = \frac{\zeta_{2n-1} - \zeta_{2n-1}}{2}, \quad 2 \leq n \leq N. \]
Here, we set additionally
\[
D_1 + \frac{\omega_1 - \omega_0}{\zeta_1 - \zeta_0} = 0, \tag{10}
\]
i.e., we set \( D_0 = 0 \), but this is not essential.

From (8) and (9) we have
\[
\frac{\partial \Phi}{\partial t} + \mathbf{CD} = 0, \tag{11}
\]
where
\[
\mathbf{C} = \mathbf{AB}. \tag{12}
\]
The equation (3), written in the geopotential levels, together with (10) form the system of \( 2N \) equations for \( \Phi_{2n-1}, D_{2n-1} \) \( (1 \leq n \leq N) \).

In the definition of the parameters \( S_s \), \( \alpha \), \( a_n \) the real temperature profile enters. Let us characterize the real profile values by index "r" i.e., \( T_r, \Gamma_r \) and respectively \( \mathbf{A}_r, \mathbf{C}_r \). In order to use the semi-implicit method, beyond the real temperature profile, the basic temperature profile should be introduced [3,12]. We shall characterize it by index "b" i.e., \( T_b, \Gamma_b \) and respectively \( \mathbf{A}_b, \mathbf{C}_b \). Further, as usually, we consider the profile \( T_b \) be independent on \( x, y \) and \( t \). Hence, for any vertical column of a model the matrices \( \mathbf{A}_b \) (and \( \mathbf{C}_b \)) are the same and constant.

According to the semi-implicit method we write the equation (10) in the form
\[
\frac{\partial \Phi}{\partial t} + \mathbf{C}_b \mathbf{D} = (\mathbf{C}_b - \mathbf{C}_r) \mathbf{D} \tag{13}
\]
and use the following second-order accurate time-integration scheme for equations (3) and (12)
\[
\frac{\mathbf{D}^{n+1} - \mathbf{D}^{n-1}}{2\tau} + \nabla^2 \left( \Phi^{n+1} - \Phi^{n-1} \right) = 0, \tag{14}
\]
where \( m \) is the time step number. Thus, an implicit method of Crank-Nicolson is used for approximation of the terms with constant coefficients in the left hand side and the other terms with variable coefficients in the right hand side are approximated (as well as, perhaps, nonlinear terms which are omitted in our analysis) explicitly by the leap-frog method [9].

### 3 Characteristic equation

Let us introduce the spectral representation of the \( N \)-dimensional vector functions \( \Phi \) and \( \mathbf{D} \)
\[
\Phi_n(x,y) = \sum_{k_1 k_2} \Phi_{k_1 k_2} e^{i(k_1 x + k_2 y)},
\]
\[
\mathbf{D}_n(x,y) = \sum_{k_1 k_2} \mathbf{D}_{k_1 k_2} e^{i(k_1 x + k_2 y)} \tag{15}
\]
where \( (k_1, k_2) \) is the wavenumber and \( \Phi_{k_1 k_2}, \mathbf{D}_{k_1 k_2} \) are the \( N \)-dimensional constant amplitudes of the wave.

According to von Neumann’s method [11], we investigate the time variation of one harmonic in (15). We substitute a wave of a wavenumber \( (k_1, k_2) \) in the equations (13) and (14) and obtain the following matrix system for amplitudes \( \Phi_{k_1 k_2}, \mathbf{D}_{k_1 k_2} \) (the indices \( k_1, k_2 \) are omitted):
\[
\frac{\lambda^2 - 1}{2\tau} \mathbf{D} - k^2 \frac{\lambda^2 + 1}{2} \mathbf{D} = 0, \tag{16}
\]
where \( k^2 = k_1^2 + k_2^2 \). If any difference approximation of space derivatives is used, the radius wavenumber \( k \) remains to be real. For example, if one uses the difference grid "c" with grid step \( h \) [9] and the central differences in space for approximation of the derivatives, then
\[
k^2 = \frac{4}{h^2} \left( \sin^2 \frac{k h}{2} + \sin^2 \frac{k h}{2} \right).
\]
Equalizing to zero the determinant of the system of the linear algebraic equations (16) for components
of the vectors \( \tilde{\Phi} \) and \( \tilde{D} \), we obtain a matrix equation for the amplification factor \( \lambda \):

\[
\begin{bmatrix}
(\lambda^2 - 1)I & -2M^2(\lambda^2 + 1)I \\
(\lambda - 1)^2C_a + \lambda C_b & (\lambda^2 - 1)I
\end{bmatrix} = 0 ,
\]

(17)

where \( I \) is the unit \( N \times N \) matrix and \( M^2 = \tau^2k^2 \).

Using a formula for block matrix determinants [7] we can rewrite the equation (17) in the form

\[
\left[(\lambda^2 - 1)^2 I + M^2(\lambda^2 + 1)\left[(\lambda - 1)^2 C_a + 2\lambda C_b\right]\right] = 0 .
\]

(18)

If the matrices \( C_a \) and \( C_b \) coincide, i.e., \( T_r = T_b \), the equation (18) reduces to the following simple form

\[
\left[(\lambda^2 - 1)^2 I + M^2(\lambda^2 + 1)^2 C_a\right] = 0 .
\]

It is obvious that the solutions of the ultimate equation are

\[
\lambda^2 = \frac{1 \pm iMC_r}{1 \pm iMC_a} ,
\]

where \( c^2 \) are the eigenvalues of the matrix \( C_a \). As it was shown by Kadyshnikov [8], in the case of \( \Gamma_b < \Gamma_d \), for the vertical approximation used in (6), (7) all the eigenvalues of the matrix \( C_a \) are positive. This proves the absolute stability of the scheme because \( |\lambda_c| = 1 \).

Now, starting from the equation (18) we shall find the stability conditions when a basic state does not coincide with a real one, i.e., \( T_r \neq T_b \).

4 Requirements to a basic state in the limit of infinite time step

First, we want to find out whether numerical stability is possible when \( M \to \infty \). One can pass to the limit \( M \to \infty \) in (18) and obtain

\[
\left[(\lambda^2 + 1)\left[(\lambda - 1)^2 C_a + 2\lambda C_b\right]\right] = 0 .
\]

(19)

First of all, we see that \( \lambda = \pm i \) are \( n \)-fold roots which do not break stability of the scheme.

As \( |\mathbf{B}| \neq 0 \), all other roots, according to (11), satisfy a matrix equation

\[
\begin{bmatrix}
(\lambda - 1)^2 A_a + 2\lambda A_b
\end{bmatrix} = 0 .
\]

Since the matrices \( A_a \) and \( A_b \) are three-cornered, we obtain from here the following very simple equation

\[
\prod_{n=1}^{N} \left[(\lambda - 1)^2 + 4\mu\lambda\right] = 0 ,
\]

(20)

where

\[
\mu_n = \left(\frac{a_r}{2d_b}\right)_n .
\]

The solutions of the equation (19) for any index \( n \) are

\[
\lambda_{1,2} = 1 - 2\mu \pm 2\left[\mu^2(1 - \mu)\right]^{1/2} .
\]

If \( \mu < 0 \) then \( |\lambda_1| > 1 \) and if \( \mu > 1 \) then \( |\lambda_2| > 1 \). When \( 0 \leq \mu \leq 1 \) we have

\[
|\lambda_{1,2}| = \left|1 - 2\mu \pm 2\left[\mu^2(1 - \mu)\right]^{1/2}\right| = 1 .
\]

Therefore the condition

\[
0 \leq \left(\frac{a_r}{2d_b}\right)_n \leq 1 , \quad 1 \leq n \leq N
\]

(22)

is necessary and sufficient for computational stability of the semi-implicit scheme when time step \( \tau \to \infty \). This condition was first obtained in [3] as sufficient one for stability in the limit \( \tau \to 0 \).

At first, let us examine a simple case when both basic and real states are polytropic, i.e.,

\[
(T_{b2n}) = (T_{b2n-1})^2 \varepsilon_{b2n}^2 \Gamma_b , \quad (T_{r2n}) = (T_{r2n-1})^2 \varepsilon_{2n}^2 \Gamma_d ,
\]

where the parameters \( \Gamma_b \) and \( \Gamma_d \) are constant. One can consider the temperatures \( T_r \) and \( T_b \) to be positive and the following conditions be fulfilled

\[
\Gamma_b < \Gamma_d , \quad \Gamma_b < \Gamma_r ,
\]

i.e., both temperature profiles are stable. In this case the condition (20) becomes

\[
\left(\varepsilon_{2n}^2 \Gamma_d - \Gamma_b \right) \leq \left(\varepsilon_{2n}^2 \Gamma_d - \Gamma_r \right) .
\]

(21)

The inequality (21) confirms both results obtained numerically by Simmons et al. [14]: the best choice of the basic state is the greatest \( (T_{b2n}) \) and the smallest \( \Gamma_b \). Under the natural restriction \( (T_{r2n}) \leq 2(T_{b2n}) \) the exact stability condition is

\[
\Gamma_b \leq \Gamma_r .
\]

(22)

In this case the inequality (21) is fulfilled for all \( \varepsilon_{2n} \).

If \( \Gamma_b > \Gamma_r \) then the power in (21) is negative and for some small values \( \varepsilon_{2n} \) the inequality (21) is violated. In particular case \( (T_{b2n}) = (T_{r2n}) \), the inequality (21) is wrong for any \( \varepsilon_{2n} \) when \( \Gamma_b > \Gamma_d \).

If the temperature profiles are not polytropic, one can apply in every model layer between \( \varepsilon_{2n} \) and \( \varepsilon_{2n+2} \) the formula

\[
\frac{T_{2n}}{T_{2n+2}} = \left(\frac{\varepsilon_{2n}}{\varepsilon_{2n+2}}\right)^{\Gamma_d/2} .
\]

Using it consequently from below up to \( \varepsilon_{2n} \) one obtain the condition (21) in the form
If a model discretization is arbitrary, the condition (22) is necessary and sufficient for fulfillment of this inequality.

5 Requirements to a basic state at any time step

Now we shall prove that if the condition (22) is not observed, the semi-implicit scheme becomes absolutely unstable for some discretization of a model by altitude.

Let us suppose that the scheme is stable and
\[ \lambda = -1 - \alpha. \]  
(23)

We substitute (23) into the \( \lambda \)-equation (18) and obtain an equation for \( \alpha \)
\[
\left( 2\alpha + \alpha^2 \right) \hat{I} + M^2 \left[ 2 + 2\alpha + \alpha^2 \right] \left[ C_b - 2(1+\alpha)C_r \right] = 0
\]
The leading coefficient of this polynomial of order \( 4N \) is
\[
\left| M^2 C_b + I \right| = \prod_{n=1}^{N} \left( 1 + M^2 e^n \right)
\]
i.e., it is positive. Then the constant term which is equal to
\[
\left| 4M^2 \left( 2C_b - C_r \right) \right| = \left( 4M^2 \right)^N \left| B \right| \left| 2A_b - A_r \right|
\]
must be positive, too. Otherwise, the product of the roots is negative and since the product of all the differences must be positive. Otherwise, some differences can be negative and if their number is odd, stability is not achieved. Thus, the more stable stratification of the basic state as compared with the real one is the necessary and sufficient condition of the stability of the semi-implicit schemes when \( \tau \to \infty \) and is the necessary condition when \( \tau \) is arbitrary. Hence, the best basic state is the isothermal one, because temperature inversions in the large-scale atmospheric processes are observed rarely. Such a choice of a basic state by no means influences a quality of computing, because the schemes of Crank-Nicolson and leap-frog have the same order of approximation, and the question is a computational stability only.

References:


