Nonlinear Decoupling Control for a Robot Manipulator

A.S. TSIRIKOS¹,² and N.E. MASTORAKIS²

¹NATIONAL TECHNICAL UNIVERSITY OF ATHENS
DIVISION OF COMPUTER SCIENCE
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING
15773 ZOGRAPHOU, ATHENS, GREECE

(²) MILITARY INSTITUTIONS OF UNIVERSITY EDUCATION
HELLENIC NAVAL ACADEMY
CHAIR OF COMPUTER SCIENCE
18539, HATZIKYRIAKOY, PIRAEUS, GREECE

Abstract: A new approach to the decoupling control problem is reported. Necessary and sufficient conditions of algebraic nature for the problem to have a solution, as well as the general analytical solution for the admissible controller are presented. Applying this approach to the robot manipulator, the general analytical expression for the decoupling control law is derived. Appropriate selection of the arbitrary parameters of the control law leads to a linearized and internally stable closed-loop system, thus simplifying the design of an additional output tracking controller.

1. Introduction

Robots have become significant tools for many industrial applications. This fact demands the development of advanced control strategies for robot manipulators. The problem of regulating manipulators is extensively studied in the literature (Paul, 1979; Saridis, 1979; Saridis, 1983; Vucobratovic, and Kircanski, 1983; Singh and Schy, 1986; Tarn, et al., 1991). One of the basic control objectives for robots is the design of prespecified trajectories for the manipulator. To meet this objective, decoupling controllers have been used in order to eliminate the coupling of motion in each direction. Paul (1979), and Vucobratovic, and Kircanski (1983), based the controller design on an approximate linearized model, around a steady state. Nonlinear controllers for robots have been proposed by Tarn, et al. (1991), and Singh, and Schy (1986). Tarm, et al (1991) investigated the effect of the motor dynamics to the equations of motion, wherein a special nonlinear decoupling controller is designed. Furthermore, the inversion and stabilisation problem for the case of elastic robots is studied by Singh, and Schy (1986).

With regard to the decoupling problem of nonlinear systems via static state feedback, many results have also been reported in the literature. The problem was first studied by Porter (1970). The first systematic results have been reported by Freund (1975), and Sinha (1977). Isidori, et al. (1981) presented a geometric approach to the decoupling control problem. In particular, necessary and sufficient criteria, of geometric nature, for the problem to have a solution are established. Moreover, a solution for the admissible controllers is proposed. A characterisation of all decoupling control laws has been reported by Ha, and Gilbert (1986), wherein the determination of the
The decoupling control law requires the solution of a homogeneous system of first order partial differential equations (whose solution, in general, is not constructable). Xia (1993) proposed a construction algorithm of the decoupling control laws. Finally, the decoupling problem with stability is studied by Isidori, and Grizzle (1988).

In this paper a new approach to the decoupling control problem under static state feedback is first presented (Tsirikos, 1996). The proposed approach reduces the determination of the desired control law to that of solving a nonhomogeneous system of first order partial differential equations, called decoupling design equations. On the basis of the decoupling design equations, the well known necessary and sufficient conditions (Isidori 1989), for the problem to have a solution, are easily rederived. Furthermore, based on the decoupling design equations, the general analytic expression of the desired control law is determined. In particular, a constructive algorithm of all the admissible controllers is presented. This algorithm requires only simple integration. It is noted that application of the general solution of the control law may lead to an internally stable closed-loop system, in case where special solutions of the control law result to unstable internal dynamics (see Ha, and Gilbert, 1986).

The foregoing decoupling technique, is subsequently applied for the control of a robot manipulator. The most general solution for the admissible decoupling control law is analytically derived. Application of this law to the robot results in a closed-loop system whose outputs (position of the wrist subsystem in each direction) may be regulated independently. The closed-loop system is nonlinear controllable and observable system and thus stabilizable (Isidori, 1989). Furthermore, it is proven that selecting appropriately the degrees of freedom of the control law, the closed-loop system becomes a linear one with arbitrarily assignable eigenvalues. Based on the linear closed-loop system, prespecified output trajectories can be easily achieved.

The results presented in this paper are part of the material reported by Tsirikos (1996).

\section{The Decoupling technique}

\subsection{Preliminaries}

Consider the nonlinear analytic system

\[
\dot{x} = g_a(x) + G(x)u, \ x(0)=x_0, \ y=h(x) \quad (1)
\]

where \(u, y \in R^m\) and the state \(x\) belongs to an open subset \(U\) of \(R^n\). The vector \(g_a(x)\), each column \(g_i(x)\) of \(G(x)\) and \(h(x)\) are analytic vector valued functions of \(x\) from \(U\) to \(R^n\) and \(R^m\), respectively.

\textbf{Definition 2.1.}

The characteristic numbers \(d_i\)'s, \(\forall i \in \{1,...,m\}\), are defined as

\[
L_{\xi_j} L_{g_{i_2}}^{d_i} h_i(x)=0, \ \forall j \in \{1,2,...,m\} \text{ and } k<d_i \\
L_{\xi_j} L_{g_{i_2}}^{d_i} h_i(x) \neq 0, \text{ for some } j \in \{1,2,...,m\},
\]

\(\forall x\) around \(x_0\), where \(L_{\tau}(\cdot)\) denotes the Lie derivative with respect to \(\tau\) (Isidori, 1989), and \(h_i(x)\) is the \(i\)-th of \(h(x)\).
2.2 The New Decoupling Technique

Definition 2.2.
A system of the form (1) is input/output (i/o) decoupled, if the i-th element of the input \( u \) affects only the i-th element of the output \( y \) for any \( x \) around \( x_0 \).

Statement of the Decoupling problem
Consider applying to system (1) the control law
\[
u = a(x) + B(x)w \quad (2)\]
where \( w \in \mathbb{R}^m \) and \( B(x) \) is nonsingular, for any \( x \) around \( x_0 \), to yield the closed-loop system
\[
\dot{x} = \tilde{g}_0(x) + \tilde{G}(x)u, \quad x(0) = x_0, \quad y_{cls} = h(x) \quad (3)
\]
where \( \tilde{g}_0(x) = g_0(x) + G(x)a(x), \tilde{G}(x) = G(x)B(x) \). The decoupling problem is defined as follows (Tsirikos, 1996): Determine a control law of the form (2) such that the resulting closed-loop system (3) is i/o decoupled. In the rest of the present Section we present the main results established by Tsirikos (1996).

Decoupling Design Equations

Theorem 2.1.
Assume that \( |B(x)| \neq 0, \forall x \) around \( x_0 \). Then, the unknown pair \( \{a(x), B(x)\} \) satisfies the following set of equations, called the Decoupling Design Equations (DDE)
\[
B^*(x) B(x) = \text{diag} \{\lambda_i(x)\}
\]
\[
\theta_i(x) \Pi_i(x) = 0
\]
\[
\xi_i(x) \Pi_i(x) = 0
\]
where
\[
\theta_i(x) = [d\phi_i(x): k_i(x): k_{i,0}(x): \ldots]
\]
\[
\xi_i(x) = [d\lambda_i(x): p_i(x): p_{i,0}(x): \ldots]
\]
\[
\Pi_i(x) = \begin{bmatrix}
[G[g_1, G] \ldots [g_m, G] [g_0, G] \ldots]
\end{bmatrix}
\]
\[
\begin{bmatrix}
dL_{s_0}^h \lambda_i h_i(x) & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & G & \ldots \\
0 & 0 & \ldots & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\]
where \( d(\cdot) \) denotes the usual gradient, \( [\tau, \sigma] \) denotes the Lie bracket operation (Isidori, 1989), \( \phi_i = L_{s_0}^{\Sigma_1} h_i, \lambda_i(x) \neq 0, \forall x \) around \( x_0 \), \( k_i, k_{i,0}, \ldots \) and \( p_i, p_{i,0}, \ldots \) depend on the Markov parameters of the closed-loop system, and
\[
B^*(x) = \Delta \begin{bmatrix}
dL_{s_0}^h \lambda_i h_i(x) \\
\vdots \\
dL_{s_0}^h \lambda_m h_m(x)
\end{bmatrix}
\]
Based on the DDE, the necessary and sufficient conditions, a special solution, the general solution for the decoupling problem as well as a constructive algorithm of all the decoupling controllers have been established. These results are given in the following.

**Necessary and Sufficient Conditions:**

**Theorem 2.2.**
The necessary and sufficient conditions for the solvability of the decoupling problem are

\[
\det [\mathbf{B}'(x)] \neq 0, \forall x \text{ around } x_0
\]

**Special Solution for the Control Law:**

A special controller pair \( \{ \hat{\mathbf{a}}(x), \hat{\mathbf{B}}(x) \} \), often called in the literature as the “standard noninteracting feedback” (see also Isidori, 1989), is

\[
\hat{\mathbf{a}}(x) = -[\mathbf{B}'(x)]^{-1} \mathbf{a}^*(x) \quad \text{and} \quad \hat{\mathbf{B}}(x) = [\mathbf{B}'(x)]^{-1}
\]

where \( \mathbf{a}^*(x) = [L_{i_0}^{d_i+1} h_i(x) \ldots L_{n_0}^{d_n+1} h_n(x)]^T \). Application of (5) to (1) leads to

\[
\dot{x} = \hat{\mathbf{g}}_0(x) + \mathbf{G}(x)u, \quad x(0) = x_0, \quad y = h(x)
\]

where \( \hat{\mathbf{g}}_0(x) = \mathbf{g}_0(x) + \mathbf{G}(x) \hat{\mathbf{a}}(x) \) and \( \mathbf{G}(x) = \mathbf{G}(x) \hat{\mathbf{B}}(x) \). The closed-loop system (9) may be separated into \( m \) subsystems of the form \( y_j^{d_i+1} = w_i \).

**General Solution for the Control Law**

**Theorem 2.3.**
The general solution for the pair \( \{ \mathbf{a}(x), \mathbf{B}(x) \} \) is given by

\[
\mathbf{a}(x) = -[\mathbf{B}'(x)]^{-1} \{ \mathbf{a}^*(x) - \phi(x) \}
\]

\[
\mathbf{B}(x) = [\mathbf{B}'(x)]^{-1} \text{diag} \{ \lambda_i(x) \}
\]

where

\[
\phi(x) = [\phi_1(x) \ldots \phi_m(x)]^T
\]

\[
\phi_i = \phi_i \begin{pmatrix} \hat{t}_{i,1}(x), \ldots, \hat{t}_{i,\sigma_i}(x), \hat{s}_{i,1}(x), \ldots, \hat{s}_{i,n-n^*}(x) \end{pmatrix}
\]

\[
\lambda_i = \lambda_i \begin{pmatrix} \hat{t}_{i,1}(x), \ldots, \hat{t}_{i,\sigma_i}(x), \hat{s}_{i,1}(x), \ldots, \hat{s}_{i,n-n^*}(x) \end{pmatrix}
\]

where \( \phi_i \) and \( \lambda_i \neq 0, \forall x \text{ around } x_0 \), are arbitrary analytic functions of their arguments.

To determine \( \hat{t}_{i,\rho} \) and \( \hat{s}_{i,\rho}, \rho = 1, \ldots, n - n^* \), the following algorithm has been proposed.

**Construction algorithm for \( \phi_i \) and \( \lambda_i \), \( i \in \{ 1, 2, \ldots, m \} \)**

**Step 1** Construct the reachability matrix
\[
\hat{Q}(x) = \begin{bmatrix}
\hat{G} & \ldots & \hat{g}_{j_1} & \ldots & \hat{g}_{j_k} & \hat{G} & \ldots \\
\end{bmatrix}
\]  
(8)

of (6), where \( j_1, \ldots, j_k \in \{0, 1, \ldots, m\} \) and \( k \in \{0, 1, \ldots\} \). The integer \( n' \) is defined by

\[
n' = \text{rank} [\hat{Q}(x)], \ \forall x \text{ around } x_0,
\]  
(9)

**Step 2** Form the matrix \( \hat{S}_i(x) \) by the columns

\[
X_k = \begin{bmatrix}
g_{j_1} & \ldots & g_{j_k} & \hat{g}_i \end{bmatrix}
\]

where \( j_1, \ldots, j_k \in \{0, 1, \ldots, m\} \) and \( k \in \{0, 1, 2, \ldots\} \). The rest of the columns of \( \hat{Q}(x) \) form the matrix \( \hat{S}_i(x) \). The number \( n_i \) is defined as the dimension of the involutive closure (Isidori, 1989) of the distribution spanned, locally around \( x_0 \), by the columns of the matrix \( \hat{S}_i(x) \), i.e.

\[
n_i = \dim \{ \text{inv} [\hat{S}_i(x)] \}, \ \forall x \text{ around } x_0
\]  
(10)

**Step 3** Define the integers \( \sigma_i = n' - n_i \).

**Step 4** Rearrange the columns of the reachability matrix \( \hat{Q}(x) \) as follows

\[
\hat{Q}(x) = \begin{bmatrix}
\hat{L}_i(x) & \hat{L}_i(x)
\end{bmatrix}
\]  
(11)

where

\[
\hat{L}_i(x) \overset{\Delta}{=} \text{inv} [\hat{S}_i(x)] : g_i(x) : \ldots : \text{ad}_{\hat{g}_i(x)}^{n_i}
\]  
(12)

**Step 5** Denote by

\[
\hat{I}_{i,1} : \ldots : \hat{I}_{i,n_i+\sigma_i+1}(x)
\]
a local base of the distribution spanned by the columns of the matrix \( \hat{L}_i(x) \).

**Step 6** Construct the matrix \( \hat{R}_i(x) \), the columns of which constitute a local base of the distribution spanned by the columns of \( \hat{Q}(x) \), as

\[
\hat{R}_i(x) = \begin{bmatrix}
\hat{I}_{i,1} : \ldots : \hat{I}_{i,n_i+\sigma_i}(x)
\end{bmatrix}
\]  
(13)
where \( \hat{I}_{1,n_i+2}(x)\ldots\hat{I}_{i,n_i+\sigma_i}(x) \) are the linear independent columns of \( \hat{L}_i \), which are linearly independent from the columns of \( L_i^* \).

**Step 7** Construct the matrix

\[
R_i^*(x) = \left[ R_i(x); \hat{d}_{i,1}(x);\ldots;\hat{d}_{i,n_i}(x) \right] \tag{14}
\]

where \( \hat{d}_{i,1}(x),\ldots,\hat{d}_{i,n_i}(x) \) are nx1 vectors orthogonal to \( R_i(x) \), such that \( d_{i,j}^T(x) \), \( j=1\ldots,n_i \), are complete differentials.

**Step 8** The functions \( t_i^*(x) \) and \( s_i^*(x) \) are the solutions of the following equations

\[
d(t_i^{\hat{\rho}}(x))=\hat{w}_{i,\hat{\rho}}(x), \quad \rho \in \{1,\ldots,\sigma_i\} \quad (15a)
\]

\[
d(s_i^{\hat{\rho}}(x))=\hat{d}_{i,\hat{\rho}}^T(x), \quad \rho \in \{1,\ldots,n-n^*\} \quad (15b)
\]

where

\[
\hat{w}_{i,\hat{\rho}}(x) = (-1)^\rho d_L^{d_i+1-\rho}h_i \quad \rho \in \{1,\ldots,d_i+1\} \tag{16}
\]

\[
\hat{w}_{i,\hat{\rho}}(x) = e_{n^*-\sigma_i+\rho}^T \left[ R_i^*(x) \right]^{-1}, \quad \rho \in \{d_i+2,\ldots,\sigma_i\}
\]

and \( en^*-\sigma_i+\rho \) denotes the \( (n^*-\sigma_i+\rho) \)-th column of \( I_n \). It is noted that the first order partial differential equations (1.5) may be solved by a simple integration.

**Remark 2.1.** If \( \sum_{i=1}^{m} (d_i+1) = n^* = n \), then the solution for \( \phi_i \) and \( \lambda_i \) is given by

\[
\phi_i = \phi_i \left( h_i,\ldots,L_{\hat{\rho}}^i h_i \right) \quad \text{and} \quad \lambda_i = \lambda_i \left( h_i,\ldots,L_{\hat{\rho}}^i h_i \right) \tag{17}
\]

**Remark 2.2.** It is proven that there exist \( \sigma_i \) linearly independent solutions for \( \phi_i \) or \( \lambda_i \) of the DDE. Furthermore, it is proven that the functions \( h_i(x),\ldots,L_{\hat{\rho}}^i h_i(x) \) are linearly independent and they are also solutions for \( \phi_i \) or \( \lambda_i \). Hence, \( \sigma_i \geq d_i+1 \).

**Remark 2.3.** If \( \sigma_i = d_i+1 \), and \( n=n^* \) then the general solution for \( \phi_i \) and \( \lambda_i \) may be immediately determined by (17).
3. The manipulator mathematical model and control problem

A typical general purpose manipulator is given in Figure 1. The state space approach is used throughout to formulate the control problem for the wrist subsystem. The dynamic equations of the motion of the wrist are expressed in angular position and angular velocity coordinates as (Saridis, 1983)

\[ \dot{x} = g_0(x) + G(x)u, \quad y = h(x) \quad (18) \]

where

\[ u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad h = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad g_0 = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix}, \quad J^{-1}(x)N(x) \]

\[ G = \begin{bmatrix} 0_{3x3} \\ J^{-1}(x) \end{bmatrix}, \quad x^T = [x_1, x_2, x_3, x_4, x_5, x_6] = [\theta_1, \theta_2, \theta_3, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3] \] is the state vector and

\[ J(x) = \begin{bmatrix} J_{11} + J_{22} \sin^2 x_2 + J_{33} \sin^2 x_3 + J_{23} \sin x_2 \sin x_3 & 0 & 0 \\ 0 & 0 & J_{23} \cos(x_2 - x_3) \\ 0 & J_{23} \cos(x_2 - x_3) & J_{33} \end{bmatrix} \]

The matrix $N(x)$ has the form

Fig. 1. Joint angles of a robot arm
\[
N(x) = \begin{bmatrix}
-\left(J_{22} \sin 2x_2 + 2J_{23} \sin x_3 \cos x_2\right)x_4x_5 \\
\frac{1}{2} J_{22} \sin x_2 + J_{23} \sin x_3 \cos x_2 x_2^2 \\
\frac{1}{2} J_{33} \sin 2x_3 + J_{23} \sin x_2 \cos x_3 x_3^2 \\
-\left(J_{33} \sin 2x_3 + 2J_{23} \sin x_2 \cos x_3\right)x_4x_6 \\
- J_{23} \sin(x_2 - x_3)x_6^2 - \omega_2 \sin x_2 \\
+ J_{23} \sin(x_2 - x_3)x_5^2 - \omega_3 \sin x_3
\end{bmatrix}
+ \begin{bmatrix}
f_1x_4 \\
f_2x_5 + f_{20} \\
f_3x_6 + f_{30}
\end{bmatrix}
\]

and
\[
F(x) = \begin{bmatrix}
0_{3x3} & I_{3x3} \\
0_{3x3} & 0_{3x3}
\end{bmatrix}
\]

In the foregoing relations, \(J_y\) are the appropriate moments of inertia, \(f_i\) are the coefficients of friction, and \(f_{i0}\) are the constants of stiction expressed constant, but all being highly dependent on the relative position of the arm. In what follows, we denote by \(U\) the subset of \(R^6\) where \(J(x)\) is invertible.

### 3.2 Control Objectives

One of most significant and difficult control problems of robots manipulators is the positioning of the wrist. This is due to the strong coupling between the motions in each direction and the nonlinearities appearing in the description of the wrist subsystem. Hence, techniques for the elimination of the coupling as well as linearization methods are appropriate for the control of the upper three joints of the robot.

### 4. The decoupling control for the robot manipulator

In this Section we apply the theoretical results presented in Section 2 to the mathematical state space model (18) of the manipulator.

#### 4.1 necessary and sufficient conditions

The characteristic numbers of (18) are \(d_1 = 1, d_2 = 1, d_3 = 1\). The matrix \(B^*(x)\) is
\[
B^*(x) = J^{-1}(x)
\]
whose determinant is different than zero for any \(x \in U\).

#### 4.2 Special solution of the control law

A special controller pair is
\[
\hat{a}(x) = \begin{bmatrix}
x_4 \delta(x_2,x_3)/2 \\
- J_{23} x_6 \cos(x_2 - x_3) \\
- J_{33} x_6 + J_{23} x_5 \cos(x_2 - x_3)
\end{bmatrix}, \hat{B}(x) = J(x)
\]
where
δ(x₂, x₃) = 
2J_{11} + J_{22} + J_{33} - J_{22} \cos 2x₂ + 2J_{23} \cos (x₂ - x₃) - J_{33} \cos 2x₃ - 2J_{23} \cos (x₂ + x₃)

Application of this controller pair to (18) results in a decoupled closed-loop system described by \( y_i^{(2)} = v_i \), for i=1,2,3. A simple study of the stability of the closed-loop system shows that the closed-loop system is not B.I.B.S. (bounded input, bounded state) asymptotically stable. It will be shown, in subsection 4.5 that by applying to (18) the general solution for \( a(x) \) and \( B(x) \) (derived in subsection 4.3), results in a B.I.B.S. asymptotically closed-loop system.

4.3 General solution of the control law

Using several algebraic manipulations, it can be proven that a basis on U of the space generated by the columns of the reachability matrix \( \hat{Q}(x) \) is given by the columns of

\[
\left[ \begin{array}{c}
\hat{g}_1 \\
\hat{g}_2 \\
\hat{g}_3
\end{array} \right]
\left[ \begin{array}{ccc}
\hat{g}_0 \cdot \hat{g}_1 & \hat{g}_0 \cdot \hat{g}_2 & \hat{g}_0 \cdot \hat{g}_3
\end{array} \right]
\]

Application of the algorithm for determining the general solution of the control law yields

\[
a(x) = \hat{a}(x) + J(x)\phi(x)
\]

\[
B(x) = J(x) \text{diag} \{ \lambda_i(x) \}
\]

where \( \phi(x) = \left[ \phi_1(h_i, L_{\theta_i}, h_i) \phi_2(h_2, L_{\theta_i}, h_2) \phi_3(h_3, L_{\theta_i}, h_3) \right]^T \) and \( \lambda_i(x) = \lambda_i(h_i, L_{\theta_i}, h_i) \), for i=1,2,3. Next, applying the general controller pair (21) to (18), results in the following closed-loop system

\[
\dot{x} = \tilde{g}_0(x) + \tilde{G}(x)w, \quad y = h(x)
\]

where

\[
\tilde{g}_0(x) = \left[ \tilde{f}_1(x) \tilde{f}_2(x) \tilde{f}_3(x) \tilde{f}_4(x) \tilde{f}_5(x) \tilde{f}_6(x) \right]^T
\]

and

\[
\tilde{G}(x) = \begin{bmatrix}
0_{3x3} \\
\vdots \\
\text{diag} \{ \lambda_i(h_i, L_{\theta_i}, h_i) \}
\end{bmatrix}
\]

where

\[
\tilde{f}_1(x) = x_4, \quad \tilde{f}_2(x) = x_5, \quad \tilde{f}_3(x) = x_6,
\]

\[
\tilde{f}_4(x) = \phi_1(h_1, L_{\theta_i}, h_i) - x_4 + 2(f_{10} - f_{1}x_4 - 2x_4x_5 \cos x_2 \gamma_1(x_2, x_3) - 2x_4x_6 \cos x_3 \gamma_2(x_2, x_3))/\delta (x_2, x_3)
\]
\[
\ddot{f}_5(x) = \phi_2(h_2, L_{s_0}h_2) - x_5 - J_{33} \sec^2(x_2 - x_3)
\]
\[
(f_{20} - f_2x_2 - \omega_2 \sin x_2 - J_{23}x_5^2 \sin(x_2 - x_1) + x_1^2 \cos x_1 \gamma_1(x_2, x_3))/J_{23}^2
\]
\[
+ \sec(x_2 - x_3)(f_{30} - f_3x_5 - \omega_2 \sin x_3 - J_{23}x_5^2 \sin(x_2 - x_3) + x_1^2 \cos x_1 \gamma_2(x_2, x_3))/J_{23}
\]
\[
\tilde{f}_6(x) = \phi_1(h_3, L_{s_0}h_3) - x_6 + \sec^2(x_2 - x_3)
\]
\[
(f_{30} - f_3x_5 - \omega_2 \sin x_2 - J_{23}x_5^2 \sin(x_2 - x_3) + x_1^2 \cos x_1 \gamma_1(x_2, x_3))/J_{23}
\]
and where
\[
\gamma_1(x_2, x_3) = J_{22} \sin x_2 + J_{23} \sin x_3
\]
\[
\gamma_2(x_2, x_3) = J_{23} \sin x_2 + J_{33} \sin x_3
\]

4.4 Structure of the closed-loop system

Since the sum \( \sum_{i=1}^{3}(d_i + 1) = 6 \) equals the dimension of the state space, the set of functions \( z_i^i(x) = L_{s_0}^k h_i(x) \), for \( 1 \leq k \leq d_i \) and \( i=1,2,3 \), defines the following local coordinates diffeomorphism, (Isidori, 1989)

\[
z = [z_1^1 z_2^1 z_3^1 z_1^2 z_2^2 z_3^2]^T = [x_1 x_2 x_3 x_4 x_5 x_6]^T
\]

In the new coordinates, the closed-loop system has been split into three decoupled subsystems having the form

\[
\begin{align*}
\dot{z}_1^i &= z_2^i \\
\dot{z}_2^i &= \phi_i(z_1^i, z_2^i) + \lambda_i(z_1^i, z_2^i)w_i \quad \text{i-th subsystem} (23)
\end{align*}
\]

Clearly, the overall system is controllable and observable.

4.5 Closed loop stability-exact linearization

In the present subsection we will show that the closed-loop system (23) can be stabilised by choosing appropriately the arbitrary functions \( \phi_i \) and \( \lambda_i \), \( i=1,2,3 \). To this end, choose the arbitrary functions \( \phi_i \) to be linear in their arguments and the functions \( \lambda_i \) to be real numbers, i.e. choose \( \phi_i(z_1^i, z_2^i) = -a_1^iz_1^i - a_2^iz_2^i \), \( \lambda_i = k_i \), where \( a_1^i, a_2^i \in R \) and \( k_i \in R-\{0\} \), for \( i=1,2,3 \). Then the closed-loop system may be decomposed into the three subsystems

\[
\begin{bmatrix}
\dot{z}_1^i \\
\dot{z}_2^i
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-a_1^i & -a_2^i
\end{bmatrix}
\begin{bmatrix}
z_1^i \\
z_2^i
\end{bmatrix} +
\begin{bmatrix}
0 \\
k_i
\end{bmatrix}w_i \quad \text{i-th subsystem} (24)
\]

\[
y_i = z_1^i
\]

for \( i=1,2,3 \), with i/o maps of the form

\[
y_i^{(2)} + a_2^iy_i^{(1)} + a_1^iy_i = k_iw_i \quad (25)
\]

Clearly, the foregoing linear form of the closed-loop system is highly desirable, since it is easy to satisfy the prescribed specifications (i.e. overshoot, stability). For example, in
order to achieve B.I.B.S. asymptotic stability for the above system, it is sufficient to choose the real constants $a_1^i, a_2^i, i=1,2,3$, to be strictly positive.

4.6 Asymptotic output tracking

With the nonlinear state equations (22) transformed into the linear form of equations (24) or the linear i/o maps (25), one can easily design output tracking controllers. To this end, denote by $y_{d,i}(t), i=1,2,3$, the prespecified output trajectories. Then the control-law $w_i = (1/k_i)\left((y_{d,i}^{(2)} + a_2^i y_{d,i}^{(1)} + a_1^i y_{d,i}^{(1)})\right)$, leads to the tracking error dynamics $e_i^{(2)} + a_2^i e_i^{(1)} + a_1^i e_i = 0$, where $e_i = y_i - y_{d,i}$, for $i=1,2,3$. If the six real positive constants $a_1^i$ and $a_2^i, i=1,2,3$, are chosen properly, then the above error dynamics may be exponentially stable.

5. Conclusions

In this paper a new approach to the decoupling control problem under static state feedback is first presented. The main feature of the proposed approach is that it reduces the determination of the admissible control laws to that of solving a nonhomogeneous system of first order partial differential equations, thus greatly facilitating the determination of the general analytic expression of the all the admissible control laws. The arbitrary parameters of the control law are given in terms of a constructive algorithm. This algorithm requires only simple integration. It is noted that application of the general solution of the control law may lead to an internally stable closed-loop system, in case where special solutions of the control law result to unstable internal dynamics.

The foregoing decoupling technique, is subsequently applied to the nonlinear model of a robot manipulator. It is proven that the decoupling problem is solvable for this robot manipulator. Moreover, all the admissible control laws are analytically derived. Application of the decoupling technique to the robot manipulator, results in a closed-loop system, whose outputs (position in each direction) can be controlled independently. The closed-loop system is controllable and observable. Finally, it is proven that appropriate choice of the degrees of freedom of the control law, leads to a linearized closed-loop system with arbitrarily assignable spectrum. Based on the linear closed-loop system, the desired signal for tracking prespecified output trajectories is designed.

Acknowledgement

The work presented in this paper has been partially funded by the Greek State Scholarship Foundation (I.K.Y.), the General Secretariat for Research and Technology of the Greek Ministry of Industry, Research and Technology and by the Siemens Hellas Co.
References: