Necessary and Sufficient Conditions for Generalized Invariant Subspaces

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Abstract

In this paper, new finite numbers of conditions which are equivalent to the so-called generalized invariant subspaces whose definitions have infinitely many conditions are studied.

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1 Introduction

In the framework of the so-called geometric approach, Bhattacharyya[1] first studied a generalized controlled \((A,B)\)-invariant subspace in order to study the parameter-insensitive disturbance-rejection problem with state feedback for uncertain linear systems. And then, Otsuka[5] studied the generalized conditioned \((C,A)\)-invariant subspace which is the dual version of generalized controlled \((A,B)\)-invariant subspace, and then the parameter-insensitive disturbance rejection problems with static output feedback and with dynamic compensator were studied[4],[5]. However, the definitions of those generalized invariant subspaces seem to be the very restrictive conditions for uncertain parameters in the sense that those definitions contain infinitely many conditions.

In this paper, new finite numbers of conditions which are equivalent to the definitions of generalized controlled \((A,B)\)-invariant subspace and generalized conditioned \((C,A)\)-invariant subspace which have infinitely many conditions, respectively are studied.

2 Preliminaries

First, we give some notations used throughout this investigation. For a linear map \(A\) from a vector space \(X\) into a vector space \(Y\) and a subspace \(\varphi\) of \(Y\) the image, the kernel, the dimension and the inverse image are denoted by \(\text{Im}(A)\), \(\text{Ker}(A)\), \(\dim(\varphi)\) and \(A^{-1}\varphi := \{x \in X \mid Ax \in \varphi\}\), respectively.

Next, consider the following linear systems defined in an Euclidean space \(X := \mathbb{R}^n\):

\[
S(\alpha, \beta, \gamma) : \begin{cases}
\frac{d}{dt} x(t) = A(\alpha)x(t) + B(\beta)u(t), \\
y(t) = C(\gamma)x(t),
\end{cases}
\]

where \(x(t) \in X\), \(u(t) \in U := \mathbb{R}^m\), \(y(t) \in Y := \mathbb{R}^r\) are the state, the input and the static (measurement) output, respectively. And coefficient matrices \(A(\alpha)\), \(B(\beta)\) and \(C(\gamma)\) have unknown parameters in the sense that

\[
A(\alpha) = A_0 + \alpha_1 A_1 + \cdots + \alpha_p A_p := A_0 + \Delta A(\alpha)
\]
\[
B(\beta) = B_0 + \beta_1 B_1 + \cdots + \beta_r B_r := B_0 + \Delta B(\beta)
\]
\[
C(\gamma) = C_0 + \gamma_1 C_1 + \cdots + \gamma_r C_r := C_0 + \Delta C(\gamma),
\]

where \(\alpha := (\alpha_1, \cdots, \alpha_p)\), \(\beta := (\beta_1, \cdots, \beta_r)\), \(\gamma := (\gamma_1, \cdots, \gamma_r)\) and \(\alpha_i \in \mathbb{R}\ (i = 1, \cdots, p), \beta_i \in \mathbb{R}\ (i = 1, \cdots, r)\) are all arbitrary real numbers.

In system \(S(\alpha, \beta, \gamma)\), \((A_0, B_0, C_0)\) and \((\Delta A(\alpha), \Delta B(\beta), \Delta C(\gamma))\) represent the nominal system model and a specific uncertain perturbation, respectively.

The definitions of generalized invariant subspaces are given as follows.
Definition 2.1

(i) $V$ is said to be a generalized controlled $(A, B)$-invariant if there exists an $F \in \mathbb{R}^{m \times n}$ such that

$$(A(\alpha) + B(\beta)F)V \subset V \quad \text{for all } (\alpha, \beta) \in \mathbb{R}^p \times \mathbb{R}^q.$$  

(ii) $V$ is said to be a generalized conditioned $(C, A)$-invariant if there exists a $G \in \mathbb{R}^{n \times \ell}$ such that

$$(A(\alpha) + GC(\gamma))V \subset V \quad \text{for all } (\alpha, \gamma) \in \mathbb{R}^p \times \mathbb{R}^r.$$  

Concerning the above generalized invariant subspaces, the following theorems have been given by papers [1] and [5], respectively.

Theorem 2.2 [1] The following three statements are equivalent.

(i) $V$ is a generalized controlled $(A, B)$-invariant.

(ii) There exists an $F \in \mathbb{R}^{m \times n}$ such that $(A_0 + B_0F)V \subset V$ and $B_0FV \subset V$ $(i = 1, \ldots, q)$, and $A_0V \subset V$ $(i = 1, \ldots, p)$.

(iii) $A_0V \subset B_0 \cap \bigcap_{i=1}^{q} B_1^{-1}V + V$ and $A_iV \subset V$ $(i = 1, \ldots, p)$. □

Theorem 2.3 [5] The following three statements are equivalent.

(i) $V$ is a generalized conditioned $(C, A)$-invariant.

(ii) There exists a $G \in \mathbb{R}^{n \times \ell}$ such that $(A_0 + GC_0)V \subset V$ and $GC_0V \subset V$ $(i = 1, \ldots, r)$, $A_0V \subset V$ $(i = 1, \ldots, p)$.

(iii) $A_0(V \cap C_0^{-1} \bigcup_{i=1}^{r} C_iV) \subset V$, $A_iV \subset V$ $(i = 1, \ldots, p)$. □

3 Main Results

This section gives very interesting results concerning generalized invariant subspaces in the sense that an infinitely many conditions is equivalent to a finite number of conditions.

Theorem 3.1

Suppose that $p_i, q_i$ $(i = 1, \ldots, p)$, $r_i, s_i$ $(i = 1, \ldots, q)$ are arbitrary fixed real numbers such that $p_i \neq q_i$ $(i = 1, \ldots, p)$ and $r_i \neq s_i$ $(i = 1, \ldots, q)$. Then, $V$ is a generalized controlled $(A, B)$-invariant if and only if there exists an $F \in \mathbb{R}^{m \times n}$ such that the following $(p + q + 1)$ invariant conditions are satisfied.

$$(A(\alpha) + B(\beta)F)V \subset V$$  \hspace{1cm} (1)

for all $(\alpha_1, \alpha_2, \ldots, \alpha_p) = (p_1, p_2, \ldots, p_p)$,

$(\beta_1, \beta_2, \ldots, \beta_q) = (r_1, r_2, \ldots, r_q)$;

$(\alpha_1, \alpha_2, \ldots, \alpha_p) = (q_1, q_2, \ldots, q_p)$,

$(\beta_1, \beta_2, \ldots, \beta_q) = (r_1, r_2, \ldots, r_q)$;

$(\alpha_1, \alpha_2, \ldots, \alpha_p) = (p_1, q_2, \ldots, q_p)$,

$(\beta_1, \beta_2, \ldots, \beta_q) = (r_1, r_2, \ldots, r_q)$;

$(\alpha_1, \alpha_2, \ldots, \alpha_p) = (p_1, q_2, \ldots, q_p)$,

$(\beta_1, \beta_2, \ldots, \beta_q) = (r_1, s_2, \ldots, s_q)$;

$(\alpha_1, \alpha_2, \ldots, \alpha_p) = (p_1, q_2, \ldots, q_p)$,

$(\beta_1, \beta_2, \ldots, \beta_q) = (r_1, s_2, \ldots, s_q)$;

$(\alpha_1, \alpha_2, \ldots, \alpha_p) = (p_1, q_2, \ldots, q_p)$,

$(\beta_1, \beta_2, \ldots, \beta_q) = (r_1, s_2, \ldots, s_q)$.

Proof. Since the proof of “Only if part” is obvious, it is suffices to show “If part”. Suppose that there exists an $F \in \mathbb{R}^{m \times n}$ such that $(p + q + 1)$ invariant conditions (1) are satisfied. Define the following two matrices as

$$\tilde{A}_0 := A_0 + \sum_{i=1}^{p} p_i A_i \quad \text{and} \quad \tilde{B}_0 := B_0 + \sum_{i=1}^{q} r_i B_i.$$  

Then, we have

$$(A(\alpha) + B(\beta)F)V = \{\tilde{A}_0 + \sum_{i=1}^{p} (\alpha_i - p_i) A_i \}
+ (\tilde{B}_0 + \sum_{i=1}^{q} (\beta_i - r_i) B_i)F\}V \subset V$$  \hspace{1cm} (2)

for all $\alpha_i \in \{p_i, q_i\}$ $(i = 1, \ldots, p)$ and $\beta_i \in \{r_i, s_i\}$ $(i = 1, \ldots, q)$.

In (1), if we choose parameters $(\alpha_1, \ldots, \alpha_p) = \ldots = (\beta_1, \ldots, \beta_q) = \ldots$.
(p_1, \ldots, p_p) and (\beta_1, \ldots, \beta_q) = (r_1, \ldots, r_q), then
\[(A(\alpha) + B(\beta)F)V = (\tilde{A}_0 + \tilde{B}_0 F)V \subset V \] (3)

In (1), if we choose parameters \((\alpha_1, \alpha_2, \ldots, \alpha_p) = (q_1, p_2, \ldots, p_p)\) and \((\beta_1, \beta_2, \ldots, \beta_q) = (r_1, r_2, \ldots, r_q)\), then

\[(A(\alpha) + B(\beta)F)V = \{\tilde{A}_0 + (q_1 - p_1)A_1 + \tilde{B}_0 F\}V \subset V \] (4)

It follows from (3), (4) and \(q_1 - p_1 \neq 0\) that
\[A_1 V \subset V.\]

Similarly, we have
\[A_i V \subset V \quad (i = 1, \ldots, p).\] (5)

Further, in (2), if we choose parameters \((\alpha_1, \alpha_2, \ldots, \alpha_p) = (p_1, p_2, \ldots, p_p)\) and \((\beta_1, \beta_2, \ldots, \beta_q) = (s_1, r_2, \ldots, r_q)\), then

\[(A(\alpha) + B(\beta)F)V = (\tilde{A}_0 + \tilde{B}_0 F + (s_1 - r_1)B_1 F)V \subset V.\] (6)

It follows from (3), (6) and \(s_1 - r_1 \neq 0\) that
\[B_1 V \subset V.\]

Similarly, we have
\[B_i V \subset V \quad (i = 1, \ldots, q).\] (7)

Now, since
\[(\tilde{A}_0 + \tilde{B}_0 F)V = (A_0 + B_0 F + \sum_{i=1}^{p} p_i A_i + \sum_{i=1}^{q} r_i B_i F)V,\]
it follows from (3), (5) and (7) that
\[(A_0 + B_0 F)V \subset V.\] (8)

Hence, (5), (7), (8) and Theorem 2.2 give that \(V\) is a generalized controlled \((A, B)\)-invariant.

**Theorem 3.2**
Suppose that \(p_i, q_i \ (i = 1, \ldots, p), \ t_i, u_i \ (i = 1, \ldots, r)\) are arbitrary fixed real numbers such that \(p_i \neq q_i \ (i = 1, \ldots, p)\) and \(t_i \neq u_i \ (i = 1, \ldots, r)\). Then, \(V\) is a generalized conditioned \((C, A)\)-invariant if and only if there exists a \(G \in \mathbb{R}^{n \times \ell}\) such that the following \((p+r+1)\) invariant conditions are satisfied.

\[(A(\alpha) + GC(\gamma))V \subset V\]

for all \((\alpha_1, \alpha_2, \ldots, \alpha_p) = (p_1, p_2, \ldots, p_p),\)
\[(\gamma_1, \gamma_2, \ldots, \gamma_r) = (t_1, t_2, \ldots, t_r);\]
\[(\alpha_1, \alpha_2, \ldots, \alpha_p) = (p_1, q_2, \ldots, p_p),\]
\[(\gamma_1, \gamma_2, \ldots, \gamma_q) = (u_1, t_2, \ldots, t_r);\]
\[(\alpha_1, \alpha_2, \ldots, \alpha_p) = (p_1, q_2, \ldots, p_p),\]
\[(\gamma_1, \gamma_2, \ldots, \gamma_q) = (t_1, t_2, \ldots, t_r); \quad \cdots;\]
\[(\alpha_1, \alpha_2, \ldots, \alpha_p) = (p_1, p_2, \ldots, q_p),\]
\[(\gamma_1, \gamma_2, \ldots, \gamma_q) = (r_1, r_2, \ldots, r_r);\]
\[(\alpha_1, \alpha_2, \ldots, \alpha_p) = (p_1, p_2, \ldots, p_p),\]
\[(\gamma_1, \gamma_2, \ldots, \gamma_q) = (u_1, u_2, \ldots, t_r);\]
\[(\alpha_1, \alpha_2, \ldots, \alpha_p) = (p_1, p_2, \ldots, p_p),\]
\[(\gamma_1, \gamma_2, \ldots, \gamma_q) = (t_1, t_2, \ldots, u_r).\]

**Proof.** The proof follows in the same manner as the proof of Theorem 3.1. 

**Remark 3.3**  Theorem 3.1 gives that an infinitely many conditions are equivalent to \((p+q+1)\) conditions. Similarly, Theorem 3.2 gives that an infinitely many conditions are equivalent to \((p+r+1)\) conditions.

The following two corollaries can be easily obtained from Theorems 3.1 and 3.2, respectively and relate to the next section.

**Corollary 3.4**
Suppose that \(p_i, q_i \ (i = 1, \ldots, p), \ r_i, s_i \ (i = 1, \ldots, q)\) are arbitrary fixed real numbers such that \(p_i \neq q_i \ (i =
1, · · · , p) and r_i ≠ s_i (i = 1, · · · , q). Then, the following four statements are equivalent.

(i) V is a generalized controlled (A, B)-invariant

(ii) There exists an \( F \in \mathbb{R}^{m \times n} \) such that

\[
(A(\alpha) + B(\beta)F)V \subset V
\]

for all \( \alpha_i \in [p_i, q_i] \) (i = 1, · · · , p) and \( \beta_i \in [r_i, s_i] \) (i = 1, · · · , q).

(iii) There exists an \( F \in \mathbb{R}^{m \times n} \) such that the following \( 2^{p+q} \) invariant conditions are satisfied.

\[
(A(\alpha) + B(\beta)F)V \subset V
\]

for all \( \alpha_i \in \{ p_i, q_i \} \) (i = 1, · · · , p) and \( \beta_i \in \{ r_i, s_i \} \) (i = 1, · · · , q).

(iv) There exists an \( F \in \mathbb{R}^{m \times n} \) such that the following \( (p + q + 1) \) invariant conditions are satisfied.

\[
(A(\alpha) + B(\beta)F)V \subset V
\]

for all \( (\alpha_1, \alpha_2, \ldots , \alpha_p) = (p_1, p_2, \ldots , p_p),\)

\[
(\beta_1, \beta_2, \ldots , \beta_q) = (r_1, r_2, \ldots , r_q);
\]

\[
(\alpha_1, \alpha_2, \ldots , \alpha_p) = (q_1, p_2, \ldots , p_p),\)

\[
(\beta_1, \beta_2, \ldots , \beta_q) = (r_1, r_2, \ldots , r_q);
\]

\[
(\alpha_1, \alpha_2, \ldots , \alpha_p) = (p_1, q_2, \ldots , p_p),\)

\[
(\beta_1, \beta_2, \ldots , \beta_q) = (r_1, r_2, \ldots , r_q);
\]

\[
(\alpha_1, \alpha_2, \ldots , \alpha_p) = (p_1, p_2, \ldots , p_p),\)

\[
(\beta_1, \beta_2, \ldots , \beta_q) = (r_1, r_2, \ldots , r_q);
\]

\[
(\alpha_1, \alpha_2, \ldots , \alpha_p) = (p_1, q_2, \ldots , p_p),\)

\[
(\beta_1, \beta_2, \ldots , \beta_q) = (r_1, s_2, \ldots , r_q);
\]

\[
(\alpha_1, \alpha_2, \ldots , \alpha_p) = (p_1, p_2, \ldots , p_p),\)

\[
(\beta_1, \beta_2, \ldots , \beta_q) = (r_1, r_2, \ldots , s_q).\]

Corollary 3.5

Suppose that \( p_i, q_i \) (i = 1, · · · , p), \( t_i, u_i \) (i = 1, · · · , r) are arbitrary fixed real numbers such that \( p_i \neq q_i \) (i = 1, · · · , p) and \( t_i \neq u_i \) (i = 1, · · · , r). Then, the following four statements are equivalent.

(i) \( V \) is a generalized conditioned \((C, A)\)-invariant.

(ii) There exists a \( G \in \mathbb{R}^{n \times \ell} \) such that

\[
(A(\alpha) + GC(\gamma))V \subset V
\]

for all \( \alpha_i \in [p_i, q_i] \) (i = 1, · · · , p) and \( \gamma_i \in [t_i, u_i] \) (i = 1, · · · , r).

(iii) There exists a \( G \in \mathbb{R}^{n \times \ell} \) such that the following \( 2^{p+q} \) invariant conditions are satisfied.

\[
(A(\alpha) + GC(\gamma))V \subset V
\]

for all \( \alpha_i \in \{ p_i, q_i \} \) (i = 1, · · · , p) and \( \gamma_i \in \{ t_i, u_i \} \) (i = 1, · · · , r).

(iv) There exists a \( G \in \mathbb{R}^{n \times \ell} \) such that the following \( (p + q + 1) \) invariant conditions are satisfied.

\[
(A(\alpha) + GC(\gamma))V \subset V
\]

for all \( (\alpha_1, \alpha_2, \ldots , \alpha_p) = (p_1, p_2, \ldots , p_p),\)

\[
(\gamma_1, \gamma_2, \ldots , \gamma_r) = (t_1, t_2, \ldots , t_r);
\]

\[
(\alpha_1, \alpha_2, \ldots , \alpha_p) = (q_1, p_2, \ldots , p_p),\)

\[
(\gamma_1, \gamma_2, \ldots , \gamma_r) = (t_1, t_2, \ldots , t_r);
\]

\[
(\alpha_1, \alpha_2, \ldots , \alpha_p) = (p_1, q_2, \ldots , p_p),\)

\[
(\gamma_1, \gamma_2, \ldots , \gamma_r) = (t_1, t_2, \ldots , t_r);\]

\[
(\alpha_1, \alpha_2, \ldots , \alpha_p) = (p_1, p_2, \ldots , p_p),\)

\[
(\gamma_1, \gamma_2, \ldots , \gamma_r) = (r_1, r_2, \ldots , r_q);
\]

\[
(\alpha_1, \alpha_2, \ldots , \alpha_p) = (p_1, q_2, \ldots , p_p),\)

\[
(\gamma_1, \gamma_2, \ldots , \gamma_r) = (r_1, r_2, \ldots , r_q);
\]

\[
(\alpha_1, \alpha_2, \ldots , \alpha_p) = (p_1, p_2, \ldots , p_p),\)

\[
(\gamma_1, \gamma_2, \ldots , \gamma_r) = (t_1, u_2, \ldots , t_r);
\]

\[
(\alpha_1, \alpha_2, \ldots , \alpha_p) = (p_1, p_2, \ldots , p_p),\)

\[
(\gamma_1, \gamma_2, \ldots , \gamma_r) = (t_1, l_2, \ldots , u_r).\]

4 Conclusions

In this paper, new finite numbers of conditions which are equivalent to the generalized invariant subspaces whose definitions have infinitely many conditions investigated by Bhattacharyya [1] and Otsuka[5] were studied.
References


