H∞ Fuzzy Control of Structural Systems
Using Takagi-Sugeno Fuzzy Model

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Abstract

This paper proposes a design method of H∞ control performance for structural systems using Tagagi-Sugeno (T-S) fuzzy model. The structural system with tuned mass damper is modeled by T-S type fuzzy model. By using parallel distributed compensation (PDC) scheme, we design a nonlinear fuzzy controller for the tuned mass damper system. This control problem will be reformulated into linear matrix inequalities (LMI) problem. Furthermore, the tuned mass damper will be designed according to the first mode of frequency of the control system and then the fuzzy controller will be found via Matlab LMI toolbox to stabilize the structural system. A simulation example is given to show the feasibility of the proposed fuzzy controller design method.

Introduction

Fuzzy control recently has increasingly attracted attention because it has been applied successfully to various nonlinear applications. Among them, the famous T-S fuzzy model was proposed by Takagi and Sugeno [1] to describe nonlinear systems. In this type fuzzy model, local dynamics in different state space regions are represented by a set of linear sub-models. The overall model of the system is then a fuzzy “blending” of these linear sub-models. Based on the T-S model, the parallel distributed compensation (PDC) concept was used to design the

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fuzzy controller of nonlinear systems [2]. In the PDC concept, the fuzzy controller shares the same fuzzy sets with the fuzzy model in the premise parts and each control rule is distributively designed for the corresponding rule of the fuzzy model. The overall nonlinear control system is not always stable even if all its subsystems are stable. Some significant stability analysis results have provided design methodologies that can ensure the overall system stability [3-6].

Recently, with the increasing research activities in the field of structural control, many control methods have been proposed [7-9]. Among these methods are the optimal control, pole placement, sliding mode control etc. However, as far as we know, the analysis of stability and stabilization problem of structural systems remains an open area. For this reason, we proposed here a fuzzy control technique as well as T-S fuzzy model to deal with structural control problem.

This organization of this paper is presented as follows: First, in order to model the structural systems, the T-S fuzzy modeling is briefly presented and the equation of motion of structural systems with tuned mass damper is constructed. Then, the $H^\infty$ stability criterion is provided for the existence of the T-S fuzzy controllers which achieve the control performance of control system via the Lyapunov theory. In this section, the control problem can be reformulated into a problem of solving linear matrix inequality (LMI). Finally, simulation results show the utility of the proposed fuzzy control methodology, and the conclusions are drawn.

**T-S Fuzzy Controller**

A nonlinear system can be approximated by a T-S fuzzy model. The T-S model consists of a set of If-Then rules. Each rule represents the local linear input-output relation of the nonlinear system and has the following form:

**A. T-S Fuzzy Model**

Plant Rule $i$: 
IF \( z_i(t) \) is \( M_{ii} \) and \( \cdots \) and \( z_g(t) \) is \( M_{ig} \)

THEN \( \dot{X}(t) = A_i X(t) + B_i U(t) + E_i \phi(t) \), \( i = 1, 2, \cdots, r \)

(1)

where \( M_{ip} \) (\( p = 1, 2, \cdots, g \)) is the fuzzy set; \( X(t) \in R^n \) is the state vector; \( U(t) \in R^m \) is the input vector; \( r \) is the rule number; \( z_i(t) \sim z_g(t) \) are the premise variables; \( A_i \in R^{n \times n} \), \( B_i \in R^{m \times n} \).

B. PDC Design

The fuzzy controller rules have the same premise parts as those of the T-S model. Linear control rule \( i \) is derived based on the state equation (1) in the consequent part of the \( i \)th model rule.

Control Rule \( i \):

IF \( z_i(t) \) is \( M_{ii} \) and \( \cdots \) and \( z_g(t) \) is \( M_{ig} \)

THEN \( U(t) = -F_i X(t) \), \( i = 1, 2, \cdots, r \)

(2)

where \( F_i \) is the local feedback gain matrix. The final control \( U \) is inferred using the Sum-Product reasoning method:

\[
U(t) = -\frac{\sum_{i=1}^{r} w_i(t) F_i X(t)}{\sum_{i=1}^{r} w_i(t)}
\]

(3)

where \( w_i \) is the activation degree of the \( i \)th rule, calculated as: \( w_i(t) = \prod_{p=1}^{g} M_{ip}(z_p) \).

Equations of Motion of Structural Systems

Assume that the equation of motion for a shear-type-building modeled by an n-degrees-of-freedom system controlled by actuators and subjected to external force \( \phi(t) \) can be characterized by the following differential equation:

\[
M \ddot{X}(t) + C \dot{X}(t) + KX(t) = BU(t) - M \ddot{\phi}(t)
\]

(4)
where \( \vec{X}(t) = [\vec{x}_1(t), \vec{x}_2(t), \ldots, \vec{x}_n(t)] \in \mathbb{R}^n \) is an \( n \)-vector; \( \ddot{X}(t), \dot{X}(t), X(t) \) are acceleration, velocity, and displacement vectors; matrices \( M, C, \) and \( K \) are \((n \times n)\) mass, damping, and stiffness matrices, respectively; \( \vec{r} \) is an \( n \)-vector denoting the influence of the external force; \( \vec{B} \) is a \((n \times m)\) matrix denoting the locations of \( m \) control forces; \( \phi(t) \) is the excitation with an upper bound \( \phi_{up}(t) \geq \|\phi(t)\| \); \( U(t) \) corresponds to the actuator forces (generated via active tendon system or an active mass damper, for example); this is only a static model.

For controller design, the standard first-order state equation corresponding to Eq.(4) is obtained by

\[
\dot{X}(t) = AX(t) + BU(t) + E\phi(t)
\]

where

\[
X(t) = \begin{bmatrix} \vec{x}(t) \\ \dot{\vec{x}}(t) \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ M^{-1}B \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ -\vec{r} \end{bmatrix},
\]

in which,

\[
A_{11} = 0, \quad A_{11} = I_{n \times n}, \quad A_{21} = \begin{bmatrix}
-k_1 + k_2 \\ -k_2 \\ k_2 + k_3 \\ -k_3 \\ \vdots \\ -k_{n-1} + k_n \\ 0 \\
-M_1 \\ -M_2 \\ -M_2 \\ -M_3 \\ \vdots \\ -M_{n-1} \\ -M_n
\end{bmatrix},
\]

\[
A_{22} = \begin{bmatrix}
0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ M_1 \\ M_2 \\ M_2 \\ M_3 \\ \vdots \\ M_{n-1} \\ M_n
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
0 \\ -M^{-1}B
\end{bmatrix},
\]

\[
E = \begin{bmatrix}
0 \\ -\vec{r}
\end{bmatrix},
\]

\[
M = \begin{bmatrix}
M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{n-1} \\ M_n
\end{bmatrix},
\]

\[
M^{-1} = \begin{bmatrix}
M_1^{-1} \\ M_2^{-1} \\ M_3^{-1} \\ \vdots \\ M_{n-1}^{-1} \\ M_n^{-1}
\end{bmatrix}
\]
For some reason (check controllable or use pole placement, for example), the constant matrix $A$ could be reformulated into canonical control form as follow.

$$A = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
a_1 & a_2 & \cdots & a_{n-1} & a_n
\end{bmatrix}. \quad \text{(6)}$$

**Fuzzy Modeling of Structural System**

To discuss the stability of the structural system in advance, Takagi-Sugeno (T-S) fuzzy models and some stability analysis are utilized to approximate the structural system.

The $i$th rule of the T-S fuzzy model, representing the structural system (5) is the following:

Rule $i$: IF $x_1(t)$ is $M_{i1}$ and \ldots and $x_p(t)$ is $M_{ip}$

THEN $\dot{X}(t) = A_iX(t) + B_iU(t) + E_i\phi(t)$ \quad \text{(7)}

By using the fuzzy inference method with a singleton fuzzifier, product inference, and center average defuzzifier, the dynamic fuzzy model (7) can be expressed as following:

$$\dot{X}(t) = \frac{\sum_{i=1}^{r} w_i(t)[A_iX(t) + B_iU(t) + E_i\phi(t)]}{\sum_{i=1}^{r} w_i(t)} \quad \text{(8)}$$

Based on Eq. (8) and the PDC described in Eq. (3), the closed-loop control system is
\[
\dot{X}(t) = \sum_{i=1}^{r} \sum_{t_i}^{t_{i+1}} h_i(t) h_i(t) [(A_i - B_i K_i) X(t)] + E_i \phi(t) 
\]

\textit{H}^{\infty} \text{ Control Design via Fuzzy Control}

In order to attenuate the influence of the excitation \( \phi(t) \) on the state variable \( X(t) \) [10, 11], this section proposes the \( H^{\infty} \) control performance. Hence, in this work, not only the stability of fuzzy control systems is advised but also the \( H^{\infty} \) control performance is satisfied as follows:

\[
\int_0^{t_f} X(t)^T Q X(t) \, dt \leq X(0)^T P X(0) + \eta^2 \int_0^{t_f} \phi(t)^T \phi(t) \, dt
\]

where \( t_f \) denotes the terminal time of the control, \( P \) are some positive definite matrices, \( \eta \) is a prescribed value which denotes the effect of \( \phi(t) \) on \( X(t) \), and \( Q \) is a positive definite weighting matrix.

\textbf{Definition} [12]: LMI Formulation of the Design Specifications

The linear matrix inequality (LMI) is any constraint of the form

\[
F(v) = F_0 + \sum_{i=1}^{m} v_i F_i \geq 0
\]

where \( v = [v_1, v_2, \cdots, v_m] \in \mathbb{R}^m \) is the variable vector, and the symmetric matrices \( F_i = F_i^T \in \mathbb{R}^{m \times m} \), \( i = 0, \cdots, m \), are given. It can be shown that the solution set \( \{v \mid F(v) > 0\} \) may be empty, but it is always convex. Thus, although (11) has no analytic solution in general, it can be solved numerically by efficient numerical algorithms. Many control problems can be reformulated into LMI’s and solved efficiently by recently developed interior-point methods [13].

A typical stability condition for fuzzy system (9) is proposed here as follows:

\textbf{Theorem 1}: The equilibrium point of fuzzy control system (9) is stable in the large if there exist
a common positive definite matrix $P$ such that the following two inequalities are satisfied:

$$
(A_i - B_i K_i)^T P + P(A_i - B_i K_i) + \frac{1}{\eta^2} P E_i E_i^T P + Q < 0
$$

(12)

with $P = P^T > 0$, for $i < l \leq r$ and $i = 1, 2, \ldots, r$

**Proof:** Using the Lyapunov function candidate for the fuzzy system (9)

$$
V = X^T(t)PX(t).
$$

(A1)

The time derivative of $V$ is

$$
\dot{V} = \dot{X}^T(t)PX(t) + X^T(t)P\dot{X}(t)
$$

$$
= \sum_{i=1}^{r} \sum_{l=1}^{r} h_i(t) h_l(t) [(A_i - B_i K_i) X(t)] + E_i \phi(t)]^T PX(t)
$$

$$
+ X^T(t) P \left\{ \sum_{i=1}^{r} \sum_{l=1}^{r} h_i(t) h_l(t) [(A_i - B_i K_i) X(t)] + E_i \phi(t) \right\}
$$

$$
= \sum_{i=1}^{r} \sum_{l=1}^{r} h_i(t) h_l(t) X^T(t) [(A_i - B_i K_i)^T P + P(A_i - B_i K_i)] X(t)
$$

$$
+ \phi^T(t) E_i^T PX(t) + X^T(t) P E_i \phi(t) - [\eta^2 \phi^T(t) \phi(t)] + \frac{1}{\eta^2} X^T(t) P E_i E_i^T PX(t)
$$

$$
+ [\eta^2 \phi^T(t) \phi(t)] + \frac{1}{\eta^2} X^T(t) P E_i E_i^T PX(t)
$$

(A2)

$$
\leq \sum_{i=1}^{r} \sum_{l=1}^{r} h_i(t) h_l(t) X^T(t) [(A_i - B_i K_i)^T P + P(A_i - B_i K_i) + \frac{1}{\eta^2} P E_i E_i^T P] X(t)
$$

$$
- \left( \frac{1}{\eta} P E_i^T X(t) - \eta \phi(t) \right)^T \left( \frac{1}{\eta} P E_i^T X(t) - \eta \phi(t) \right) + \eta^2 \| \phi_{ap}(t) \|^2.
$$

(A3)

Based on Theorem 1 and (A4),

$$
\dot{V} \leq \sum_{i=1}^{r} \sum_{l=1}^{r} h_i(t) h_l(t) X^T(t) \{-Q\} X(t) + \eta^2 \| \phi_{ap}(t) \|^2
$$

$$
= -X^T(t) Q X(t) + \eta^2 \| \phi_{ap}(t) \|^2
$$

(A4)
Integrating (A5) from \( t = 0 \) to \( t = t_f \) yields

\[
V(t_f) - V(0) \leq -\int_0^{t_f} X(t)^T QX(t) dt + \eta^2 \int_0^{t_f} \phi(t)^T \phi(t) dt .
\]  

(A6)

From (A1), we get

\[
\int_0^{t_f} X(t)^T QX(t) dt \leq X^T(0) PX(0) + \eta^2 \int_0^{t_f} \phi(t)^T \phi(t) dt .
\]  

(A7)

Eq. (A7) is Eq. (10). Therefore, the \( H^\infty \) control performance is achieved.

**Lemma 1** [13, 14]: (Schur Complements)

The LMI

\[
\begin{bmatrix}
Q(x) & S(x) \\
S(x) & R(x)
\end{bmatrix} > 0
\]

(14)

where \( Q(x) = Q^T(x) \), \( R(x) = R^T(x) \) and \( S(x) \) depends on \( x \) is equivalent to

\[
R(x) > 0, \quad Q(x) - S(x)R^{-1}(x)S^T(x) > 0
\]

(15)

In other words, the set of nonlinear inequalities (15) can be represented as the LMI (14).

**Remark 1**: Theorem 1 can be reformulated into the linear matrix inequality (LMI) problem and efficient interior-point algorithms are now available in Matlab toolbox to solve this problem. Therefore, Eq. (12-13) is transformed to the LMI by the following procedure.

By introducing new variables \( H_{i\ell} = A_i - B_i K_{i\ell}, \) \( W = P^{-1}, \) and \( Y_{i\ell} = H_{i\ell} W . \) Eqs. (12) can be rewritten as follows by Lemma 1:

\[
\begin{bmatrix}
Y_{i\ell} + Y_{i\ell}^T + \frac{1}{\eta^2} E_i E_i^T \\
W
\end{bmatrix}
\begin{bmatrix}
W \\
- Q^{-1}
\end{bmatrix} < 0
\]

(A6)

**A Simulation Example**

The applications of the model-based fuzzy control methods presented in this paper are illustrated in this section. The four-story building is considered. The mass, stiffness of each floor
mass is 345600 kg, 3.1*10^8 nt/m, and damping ratio is 0.02. The E-W component of the Taiwan Chi Chi earthquake record in 1999 scaled to a maximum ground acceleration of 1g is used as input excitation shown in Fig. 1, and its duration is 35 seconds. First, establish T-S fuzzy models for structural systems. Then, design a model-based fuzzy controller via the concept of PDC scheme. If there exist a positive definite matrix $P$ and feedback gain $K$ to satisfy the stability condition. Table 1 shows the maximum response of the first, second, third, fourth floor with and without input control. The displacement time histories of the 1st, 2nd, 3rd, and 4th floor with and without model-based control are presented in Figs. 2-5 and detailed system parameters are in appendix. Simultaneously, some system parameters including common definite P, feedback gain, and membership functions, are described in Appendix.

**Conclusions**

This paper is concerned discusses the stability problem of the structural systems presented by T-S type fuzzy model. In the design procedure, we represent the fuzzy system as a family of local state space models, and construct a global fuzzy logic controller by blending all such local state feedback controllers. A stability criterion of $H^\infty$ control performance is derived for the fuzzy system based on Lyapunov theory. Based on this criterion, the fuzzy controller design problem can be reduced into LMI problem. The effectiveness and the feasibility of the proposed controller design method is demonstrated through numerical simulations on the four-story shear building under a seismic excitation, Taiwan Chi Chi earthquake occurred in 1999. This proposed methodology could be applied in practical structural system from the example.
No control
\( U_{\text{max}} = 0 \text{ kn} \)

Model-based control
\( U_{\text{max}} = 4.161 \text{ kn} \)

<table>
<thead>
<tr>
<th>( x_j ) (m)</th>
<th>( \ddot{x}_j ) (m/s(^2))</th>
<th>( x_j ) (m)</th>
<th>( \ddot{x}_j ) (m/s(^2))</th>
</tr>
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<tr>
<td>1f</td>
<td>0.0175</td>
<td>10.75</td>
<td>0.0072</td>
</tr>
<tr>
<td>2f</td>
<td>0.027</td>
<td>13.21</td>
<td>0.0139</td>
</tr>
<tr>
<td>3f</td>
<td>0.0361</td>
<td>10.81</td>
<td>0.0197</td>
</tr>
<tr>
<td>4f</td>
<td>0.0409</td>
<td>14.75</td>
<td>0.0129</td>
</tr>
</tbody>
</table>

Table 1. Maximum response with and without input control.

Fig. 1. The Chi Chi earthquake.

Fig. 2. Time histories of response quantities of the first floor.
Fig. 3. Time histories of response quantities of the second floor.

Fig. 4. Time histories of response quantities of the 3rd floor.

Fig. 5. Time histories of response quantities of the 4th floor.
Appendix

The nature frequency of the 4-story building, \( w_s (Hz) \Rightarrow \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 1.65 \\ 4.77 \\ 7.31 \\ 8.96 \end{bmatrix} \)

\( m_d = 20736 \text{ kg, } \ k_d = 2240000 \text{ N/m, damping ratio of TMD = 0.02.} \)

The nature frequency of the building with TMD, \( w_s (Hz) \Rightarrow \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} = \begin{bmatrix} 1.53 \\ 1.79 \\ 4.78 \\ 7.31 \\ 8.96 \end{bmatrix} \)

\[
A = 10^3 \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0.001 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.001 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.001 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.001 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.001 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.001 \\
-1.794 & 0.897 & 0 & 0 & 0 & -0.002 & 0.0012 \\
0.897 & -1.794 & 0.897 & 0 & 0 & 0.0012 & -0.002 & 0.0012 \\
0 & 0.897 & -1.794 & 0.897 & 0 & 0.0012 & -0.002 & 0.0012 \\
0 & 0 & 0.897 & -0.904 & 0.0065 & 0.0012 & -0.0012 & -0 \\
0 & 0 & 0 & 0.108 & -0.108 & 0.0004 & -0.0004 & -0 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -0.0289 & 0.4823 \end{bmatrix}^T \times 10^{-4}.
\]
$$K = \begin{bmatrix}
-1.9115e+014 & 1.4386e+015 & -8.9562e+014 & -1.0984e+012 & 7.8393e+007 & 8.5543e+014 \\
-4.2719e+014 & -1.962e+012 & 1.7109e+009 & 2.0611e+008 & & \\
\end{bmatrix}$$

$$P = \begin{bmatrix}
7.6093e+008 & -4.4515e+008 & 4.2556e+007 & -39485 & 37.401 & 1.3773e+006 \\
-4.4515e+008 & 3.158e+008 & -6.1976e+007 & 58560 & -16.331 & 2.6176e+007 \\
4.2556e+007 & -6.1976e+007 & 2.7205e+007 & -26364 & -1.689 & -1.797e+007 \\
-39485 & 58560 & -26364 & 338.47 & 0.001634 & 18869 \\
37.401 & -16.331 & -1.689 & 0.001634 & 1.2734e-005 & 1.5828 \\
1.3773e+006 & 2.6176e+007 & -1.797e+007 & 18869 & 1.5828 & 1.8275e+007 \\
-1.4242e+006 & -1.2712e+007 & 8.9802e+006 & -9222.5 & -0.84695 & -9.159e+006 \\
89358 & -99565 & 36866 & -181.22 & -0.0014039 & -19581 \\
1510.6 & -645.87 & -76.046 & 0.075851 & 0.00022322 & 73.463 \\
91.275 & -39.49 & -4.284 & 0.0053431 & 1.3364e-005 & 4.2834 \\
\end{bmatrix}$$

Columns 7 through 10

$$\begin{bmatrix}
-1.4242e+006 & 89358 & 1510.6 & 91.275 \\
-1.2712e+007 & -99565 & -645.87 & -39.49 \\
8.9802e+006 & 36866 & -76.046 & -4.284 \\
-9222.5 & -181.22 & 0.075851 & 0.0053431 \\
-0.84695 & -0.0014039 & 0.00022322 & 1.3364e-005 \\
-9.159e+006 & -19581 & 73.463 & 4.2834 \\
4.5918e+006 & 9562 & -39.575 & -2.3129 \\
9562 & 191.18 & -0.085303 & -0.0043695 \\
-39.575 & -0.085303 & 0.016309 & 0.00097648 \\
-2.3129 & -0.0043695 & 0.00097648 & 5.8481e-005 \\
\end{bmatrix}$$
References


