Optimal Control in Superpotential for Evolution 
Hemivariational Inequality *

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Abstract: In this paper we study the optimal control of system driven by hemivariational inequality of second order. First, we establish the existence of solutions to hemivariational inequality which contains nonlinear pseudomonotone evolution operator. Introducing a control variable in the multivalued term of the generalized subdifferential, we prove the closedness (in suitable topologies) of the graph of the solution map. Then we use this result and the direct method of the calculus of variations to show the existence of optimal admissible state–control pairs.

Key–Words: Hemivariational inequality, pseudomonotone operator, multifunction, optimal control problem, Clarke’s subdifferential, constitutive law.

1 Introduction

In this paper we study the optimal control problem for system governed by the second order evolution hemivariational inequality. By a hemivariational inequality we mean an evolution inclusion involving a nonmonotone and multivalued subdifferential mapping. The motivation of our study comes from mechanics and engineering science where several problems with nonconvex superpotentials have as variational formulations hemivariational inequalities, for example in unilateral contact problems in nonlinear elasticity, in problems describing the adhesive and frictional effects, in nonconvex semipermeability problems, in masonry structures, delamination problems in multilayered composites, see e.g. [8] for detailed descriptions.

A large class of mechanical laws between stresses and strains, between boundary displacements (or velocities) and reactions, or generally between forces and fluxes can not be modelled by convex superpotentials (see [10], [12]). In order to develop a mathematical theory to deal with such laws, P.D. Panagiotopoulos introduced in [12] the nonconvex superpotentials with the help of the generalized Clarke gradient. In this case the corresponding variational formulations are hemivariational inequalities.

The aim of this paper is twofold: to give the result on existence of weak solutions for the hemivariational inequality and next to study the optimal control problem, the state of which is described by this inclusion.

We would like to underline that our result is applicable to hemivariational inequalities for multidimensional superpotential laws, i.e. for hemivariational inequalities on vector-valued function spaces (the hyperbolic hemivariational inequalities studied in [11] and [13] admit only one-dimensional laws).

The optimal control problems of systems governed by hemivariational inequalities are nonclassical ones since the state of the problem (which is not uniquely determined) is connected with the control through a hemivariational inequality. We recall that there is a vast literature on optimal control distributed parame-
ter systems for partial differential equations (see e.g. [4]) and for control problems for variational inequalities (cf. [2]). To our knowledge there has been very little effort towards mathematical studies in the context of control of hemivariational inequalities. With respect to all the aforementioned mechanical laws the optimal control problems have a great importance.

We consider the control problem in which the control appears in the multivalued term. Such a situation corresponds to the parameter identification (inverse) problem when the goal is to identify unknown parameters responsible for the multivalued law (see e.g. the "equivalent diagram" for a real stress-strain law in composite materials in [14]). For the control problem we give sufficient conditions on the existence of optimal solutions. The important point is the variable assigns the solution set of hemivariational inequality (see Lemma 2).

The results concerning optimal control problems obtained in this work generalize the earlier ones of Miettinen and Haslinger (see [5]), Migórski and Ochal (cf. [6]) who considered the stationary hemivariational inequality, and of Migórski and Ochal (cf. [7]) for the parabolic hemivariational inequality.

The outline of the paper is as follows. In Section 2 we briefly formulate the hemivariational inequality and give the existence result. We end with giving some concluding remarks and open problems in Section 4.

2 Evolution Hemivariational Inequalities of Second Order

The goal of this section is to formulate evolution hemivariational inequalities and to prove the existence of solutions to such problems. We start from giving some notations which are used in the sequel.

Let $V$ and $X$ be two reflexive, separable Banach spaces, and let $H$ be a Hilbert space. Suppose that

$$V \subset X \subset H \approx H^* \subset X^* \subset V^*,$$

where $H^*$, $X^*$ and $V^*$ denote dual spaces to $H$, $X$ and $V$, respectively, all embeddings are dense and continuous, and $V$ embeds compactly in $X$. We denote by $(\cdot, \cdot)$ the duality of $V$ and $V^*$ and the pairing between $X$ and $X^*$ as well, by $\| \cdot \|_E$ the norm in the space $E$ being respectively $V, X, X^*$ or $V^*$ and by $| \cdot |_H$ the norm in $H$. Moreover, the symbol $w - E$ stands for the space $E$ equipped with the weak topology.

To have a concrete example, let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Typically $V = W^{1,p}(\Omega; \mathbb{R}^N)$ or $V = W_0^{1,p}(\Omega; \mathbb{R}^N)$, $X = L^p(\Omega; \mathbb{R}^N)$ and $H = L^2(\Omega; \mathbb{R}^N)$ with some $2 \leq p < \infty$. From the Sobolev embedding theorem (cf. e.g. [16]), we get that $(V, X, H, X^*, V^*)$ is an evolution fivefold and $V$ embeds compactly in $X$.

Given a fixed number $0 < T < +\infty$ we introduce the following function spaces $\mathcal{V} = L^p(0,T;V)$, $X = L^p(0,T;X)$, $\mathcal{H} = L^2(0,T;H)$, $X^* = L^q(0,T;X^*)$, $V^* = L^q(0,T;V^*)$ for some $2 \leq p < \infty$, $1/p + 1/q = 1$ and

$$\mathcal{W} = \{ w \in \mathcal{V} : w' \in \mathcal{V}^* \},$$

where the time derivative is understood in the sense of vector valued distributions. The latter is a separable, reflexive Banach space with the norm $\| w \|_{\mathcal{W}} = \| w \|_V + \| w' \|_{\mathcal{V}^*}$. Clearly we have

$$\mathcal{W} \subset \mathcal{V} \subset \mathcal{X} \subset \mathcal{H} \subset X^* \subset \mathcal{V}^*$$

with dense and continuous embeddings. Since we have assumed that $V \subset X$ compactly, we have also that $\mathcal{W} \subset \mathcal{X}$ compactly. Moreover, the embedding $\mathcal{W} \subset C(0,T;H)$ is continuous. So every equivalence class in $\mathcal{W}$ has a unique representative in $C(0,T;H)$. We also introduce the space

$$\mathcal{Z} = \{ w \in \mathcal{V} : w' \in \mathcal{W} \},$$

which is a separable, reflexive Banach space with the norm $\| w \|_{\mathcal{Z}} = \| w \|_V + \| w' \|_{\mathcal{W}}$. We say that $\{ w, w_n \} \subseteq \mathcal{Z}$, $w_n \rightarrow w$ weakly in $\mathcal{Z}$ if and only if $w_n \rightarrow w$ weakly in $\mathcal{V}$ and $w'_n \rightarrow w'$ weakly in $\mathcal{W}$. ...
in $W$. Moreover, each function $w \in Z$ is an absolutely continuous function from $[0, T]$ to $V$.

We are going to deliver an existence result for the following initial value problem. Find $y \in Z$ such that

$$\begin{cases}
y''(t) + A(t, y(t)) + By(t) + \partial J(t, y(t)) \ni f(t) \\
y(0) = y_0, \quad y'(0) = y_1,
\end{cases}$$

(1)

where the hypotheses on the data are the following:

**H(A):** $A: (0, T) \times V \to V^*$ is such that

(i) $A(\cdot, v)$ is measurable on $(0, T)$, $\forall v \in V$;

(ii) $\forall t \in (0, T)$, $A(t, \cdot) \in V^*$ is pseudomonotone;

(iii) $A(t, \cdot)$ is bounded for $t \in (0, T)$, i.e., $\|A(t, v)\|_{V^*} \leq a_1(t) + b_1\|v\|_{V}^{p-1}$, $\forall v \in V$, a.e. $t \in (0, T)$ with $a_1 \in L^1(0, T)$, $\beta_1 \geq 0$ and $b_1 > 0$;

(iv) $A(t, \cdot)$ is coercive for $t \in (0, T)$, i.e., $\langle A(t, v), v \rangle_{V^* \times V} \geq \beta_t\|v\|_{V}^{p} - \beta_2\|v\|_{V}^p - a(t)$, $\forall v \in V$ and a.e. $t \in (0, T)$, where $\beta_1 > 0$, $\beta_2 \geq 0$,

$\exists$ a.e. $a \in L^1(0, T)$ and $r < p$.

**H(B):** $B: V \to V^*$ is a bounded, linear, monotone and symmetric operator.

**H(J):** $J: (0, T) \times X \to \mathbb{R}$ is such that

(i) $J(\cdot, x)$ is measurable on $(0, T)$, $\forall x \in X$;

(ii) $\forall t \in (0, T)$, $J(t, \cdot)$ is locally Lipschitz on $X$;

(iii) $\exists \sigma > 0$, $\forall x \in X$, $t \in (0, T)$ we have $\zeta \in \partial J(t, x) \Rightarrow \|\zeta\|_{X^*} \leq \sigma(1 + \|x\|_{X}^{2/q})$.

**H(0):** $f \in V^*$, $y_0 \in V$, $y_1 \in H$.

**H(1):** If $p = 2$ then $\frac{\beta_1}{2} > \sigma \beta^2 T$, where $\beta$ is an embedding constant of $V$ into $X$.

In the hypothesis $H(J)$ we use the notation $\partial J$ for Clarke’s gradient of $J$ with respect to the second variable $x$. Moreover, let us notice that the initial conditions in the problem (1) have a sense since the embeddings $Z \subset C(0, T; V)$ and $W \subset C(0, T; H)$ are continuous.

A function $y \in V$ is called a solution to the problem (1) if and only if $y \in W$ and $y''(t) + A(t, y(t)) + By(t) + \zeta(t) = f(t)$ for a.e. $t \in (0, T)$, $y(0) = y_0$, $y'(0) = y_1$, where $\zeta \in X^*$ is a selection such that $\zeta(t) \in \partial J(t, y(t))$ for a.e. $t \in (0, T)$.

The above hypotheses guarantee the following existence result.

**Theorem 1** If hypotheses $H(A)$, $H(B)$, $H(J)$, $(H_0)$ and $(H_1)$ hold, then the problem (1) has at least one solution.

**Sketch of proof.** We present only the main ideas of the proof of Theorem 1. For details we refer to [9]. The proof consists essentially of three steps. Firstly, we suppose a regular data $y_1 \in V$ and by using an integral operator we reduce our problem to an evolution inclusion of first order. Next, by employing the surjectivity result of [15], Theorem 2.1, we obtain the existence of solutions of the first order problem. The crucial point here is to establish the $L$-generalized pseudomonotonicity of a corresponding evolution operator. Finally, by a density argument we remove the restriction $y_1 \in V$ and prove the result for $y_1 \in H$. In that part of proof a priori estimates for solutions are essential.

**3 Control in Superpotential**

In this section we consider an important class of optimal control problems for hemivariational inequalities in which the control appears in the multivalued subdifferential term. Such control problems are more complicated than the ones for partial differential equations and variational inequalities (see e.g. [4], [2]) since now the state of the problem is related to the control parameter through a hemivariational inequality. The framework is quite general and covers in particular the parameter identification (inverse) problems for systems governed by hemivariational inequalities (cf. [14], [3]). The main theorem of this section generalizes the result of the paper of Miettinen and Haslinger ([5]) who considered the stationary hemivariational inequality with one-dimensional superpotential law. We refer also to [10] and [8] for some applications of our results to engineering structures.

We begin with a system described by the se-
cond order evolution inclusion
\begin{align*}
\begin{cases}
y''(t) + A(t, y'(t)) + B_y(t) + \partial J_u(t, y(t)) &\geq f(t) \\
y(0) = y_0, \quad y'(0) = y_1.
\end{cases}
\end{align*}
(2)

Here $A, B, f, y_0, y_1$ are as in Section 2. The family of functionals $J_u$, parameterized by controls $u$, satisfies $H(J)$ uniformly with respect to $u$ and $y$. Moreover we admit the following hypotheses.

$H(U)$: the class of admissible controls $U_{ad}$ is a compact subset of a metric space of controls $U$.

$H(J)_1$: for any $u \in U$, $J_u: (0, T) \times X \to \mathbb{R}$ satisfies $H(J)$ uniformly with respect to $u$ and $y$.

if $u_n \to u$ in $U$, then for a.e. $t \in (0, T)$

\[
\begin{align*}
\limsup_{n \to +\infty} \sup_{t, y} \partial J_{u_n}(t, \cdot) \subset \text{graph} \partial J_u(t, \cdot).
\end{align*}
\]
(3)

Here $\limsup_{n \to +\infty} A_n$ denotes the upper limit of sets in the sense of Kuratowski (in the suitable topologies).

Let $S(u)$ denote the set of solutions to (2) corresponding to a control $u$. We consider the solution map

\[
S: U \ni u \mapsto S(u) \subseteq Z.
\]

The formulation of the optimal control problem (CP) is as follows:

Given a cost functional

\[
F: U \times Z \to \mathbb{R},
\]

find $u^* \in U_{ad}$ and $y^* \in S(u^*)$ such that

\[
F(u^*, y^*) \leq F(u, y), \quad \forall u \in U_{ad}, y \in S(u).
\]

We recall that an admissible state-control pair $(y, u)$ for (CP) is a pair of a state function $y \in S(u)$ and a control $u \in U_{ad}$. By an optimal solution to (CP) we understood any admissible pair $(y^*, u^*)$ such that $F(u^*, y^*) \leq F(u, y)$ for all admissible pairs $(y, u)$.

Our goal is to present a theorem on the existence of solutions to (CP). The crucial property in studying such a problem is to establish the closedness (in suitable topologies) of the graph of the solution map.

**Lemma 2** If the hypotheses $H(A)$, $H(B)$, $H(J)_1$, $(H_0)$, $(H_1)$ and $H(U)$ hold, then the solution map $S: U \ni u \mapsto S(u) \in 2^{Z} \setminus \{\emptyset\}$ has a closed graph in $U \times (w-Z)$-topology.

**Proof.** First we observe that according to Theorem 1 for every fixed $u \in U$ the set $S(u)$ is nonempty. Let $\{u_n\} \subseteq U$ and $u_n \to u$ in $U$, $y_n \in S(u_n)$, $y_n \to y$ weakly in $Z$. Thus for every $n \in \mathbb{N}$, we have

\[
\begin{align*}
\begin{cases}
y''_n(t) + A(t, y'_n(t)) + B y_n(t) + \zeta_n(t) &\geq f(t) \quad \text{for a.e. } t \in (0, T) \\
y_n(0) = y_0, \quad y'_n(0) = y_1 \quad \zeta_n(t) \in \partial J_{u_n}(t, y_n(t)) \quad \text{for a.e. } t \in (0, T).
\end{cases}
\end{align*}
\]
(4)

Since $y_n \to y$ weakly in $W$ and $W \subset X$ compactly, so for a subsequence, we get $y_n \to y$ in $X$, and next $y_n(t) \to y(t)$ in $X$ for a.e. $t \in (0, T)$. Hence the hypothesis $H(J)_1(3)$ implies that for a.e. $t \in (0, T)$

\[
\begin{align*}
(w-X^*)-\limsup_{n \to +\infty} \partial J_{u_n}(t, y_n(t)) \subset \partial J_u(t, y(t)).
\end{align*}
\]
(5)

Indeed, for a fixed $t \in (0, T)$ and for every $z \in (w-X^*)-\limsup_{n \to +\infty} \partial J_{u_n}(t, y_n(t))$ there exists a subsequence $\{z_{n_k}\}$, $z_{n_k} \in \partial J_{u_{n_k}}(t, y_{n_k}(t))$, $z_{n_k} \to z$ weakly in $X^*$. Hence

\[
(y(t), z) \in (w-X^*)-\limsup_{n \to +\infty} \partial J_{u_n}(t, \cdot) \subset \text{graph} \partial J_u(t, \cdot),
\]

which means that $z \in \partial J_u(t, y(t))$.

Now, due to the uniform in $u$ bound in $H(J)(iii)$ and since $\{y_n\}$ is bounded in $V \subset X$, we get that $\{z_n\}$ is bounded in a reflexive Banach space $X^*$, so for a subsequence (if necessary), we have $z_n \to z$ weakly in $X^*$ with some $z \in X^*$. Applying the Convergence Theorem (see [1]) and from (5), we obtain for a.e. $t \in (0, T)$ that

\[
\begin{align*}
\zeta(t) \in \overline{\text{co}} (w-X^*)-\limsup_{n \to +\infty} \{\zeta_n(t)\}
\subset \overline{\text{co}} (w-X^*)-\limsup_{n \to +\infty} \partial J_{u_n}(t, y_n(t))
\subset \overline{\text{co}} \partial J_u(t, y(t)) = \partial J_u(t, y(t)).
\end{align*}
\]
(6)

Analogously as in the proof of Theorem 1 (see [9]), we have $A y'_n \to A y'$ weakly in $V^*$, where
\( A \) is the Nemyckii operator corresponding to \( A \). Hence and since \( y_n'' \rightharpoonup y'' \) weakly in \( \mathcal{V}^* \), \( B \) is weakly continuous, passing to the limit in (4), we obtain \( y'' + Ay' + By + \zeta = f \). To examine the initial conditions, let us notice that the weakly convergence in \( \mathcal{W} \) implies the (pointwise) weakly convergence in \( H \) for all \( t \in [0, T] \). So we get for all \( t \in [0, T] \)

\[
y_n(t) \rightharpoonup y(t) \text{ weakly in } H, \\
y'_n(t) \rightharpoonup y'(t) \text{ weakly in } H,
\]

which implies that

\[
y_0 = y_n(0) \rightharpoonup y(0) \text{ weakly in } H,
\]

so \( y(0) = y_0 \), and analogously

\[
y_1 = y'_n(0) \rightharpoonup y'(0) \text{ weakly in } H,
\]

which shows that \( y'(0) = y_1 \). Hence and from (6) finally we get \( y \in S(u) \).

Now we are in a position to prove the existence result of the control problem (CP).

**Theorem 3** If the hypotheses \( H(A), H(B), H(J)_1, (H_0), (H_1) \) and \( H(U) \) hold, the cost functional \( F \) is lower semicontinuous in \( U \times (w-\mathcal{Z}) \)-topology, then the problem (CP) admits an optimal solution.

**Proof.** By Theorem 1 we have \( S(u) \neq \emptyset \) for all fixed \( u \in U \). Let \( \{ y_n, u_n \} \subseteq U \times Z \) be a minimizing sequence for the problem (CP), i.e. \( u_n \in U_{ad}, y_n \in S(u_n) \) and

\[
\lim_{n \to +\infty} F(u_n, y_n) = \inf_{u \in U_{ad}, y \in S(u)} F(u, y) =: m.
\]

Since \( U_{ad} \) is a compact subset in \( U \), we may assume that

\[
u_n \rightharpoonup u^* \text{ in } U \text{ with } u^* \in U_{ad}.
\]

It can be shown that from the uniform in \( u \) bound in \( H(J)(iii) \), we have

\[
\|y_n\|_{C(0,T;\mathcal{V})} + \|y'_n\|_{\mathcal{W}} \\
\leq C \left(1 + \|y_0\|^2_{\mathcal{V}} + \|y_1\|^2_H + \|f\|^2_{Y^*}\right).
\]

Hence we immediately get that \( \{y_n\} \) and \( \{y'_n\} \) are uniformly bounded in \( \mathcal{V} \) and \( \mathcal{W} \), respectively. So for a subsequence we have \( y_n \rightharpoonup y^* \) weakly in \( Z \). Hence and from (7), applying Lemma 2, we obtain \( y^* \in S(u^*) \). Finally due to the lower semicontinuity of \( F \), we have

\[
m \leq F(u^*, y^*) \leq \liminf_{n \to +\infty} F(u_n, y_n) = m,
\]

which completes the proof. ◦

**4 Concluding Remarks**

- The control problems with hemivariational inequalities as the state equations are not well posed in the sense of Hadamard since in general the solution set contains more than one element.
- For the unicity of solutions we need some additional hypotheses, like, for instance, strict convexity of superpotentials and maximal monotonicity of the involved operators.
- The sufficient conditions for the convergence in (3) of Clarke’s generalized gradients have been found by Zolezzi [17]. Namely, if the sequence \( \{J_u\}_{u \in U_{ad}} \) is epi-convergent, locally equi-bounded and equi-lower semidifferentiable, then (3) holds (see Theorem 1 in [17], p.384). However, the problem how the superpotentials depend on the control variable, in the way the condition (3) holds, is still under investigation.
- The mentioned in Introduction "equivalent diagram" relates to nonmonotone multivalued stress–strain laws in composite materials (cf. [14]). Assume that the behaviour of the material of a structure has been experimentally investigated and described by a complicated sawtooth stress–strain diagram.

\[\text{Diagram}\]
Our task is to find the best possible simple fictitious stress–strain diagram, which describes the behaviour of the considered structure under the given loading. For instance, the vertices of the line describing the fictitious stress–strain law are defined by the pairs \{u_σ, u_ε\} \{\{a strain, a stress\}\} for a onedimensional stress–strain law, and their vector \( u = (u_σ, u_ε) \) is considered as a control parameter.

References: