A Closed-form Discrete Fractional Gabor Expansion

EROL ÖNEN and AYDIN AKAN[1]
Department of Electrical and Electronics Engineering
Istanbul University
Avcilar, 34850 Istanbul
TURKEY

Abstract: - We present a discrete fractional Gabor expansion based on the closed form discrete fractional Fourier transform. The traditional Gabor expansion represents a signal as a linear combination of time and frequency shifted basis functions. This constant-bandwidth analysis generates a rectangular time-frequency lattice which might lead to poor time-frequency localization for many signals. Proposed expansion uses a set of basis functions related to the fractional Fourier basis and generate a parallelogram-shaped tiling. Completeness and biorthogonality conditions of the new expansion are given.

Key-Words: - Gabor expansion, Time-frequency analysis, Fractional Fourier transform.

1 Introduction

Time-frequency (TF) analysis provides a distribution of signal energy over the joint time-frequency plane [1,2]. One of the fundamental issues in the TF analysis is obtaining the distribution of signal energy over joint TF plane with sufficient time and frequency resolution (ideally with delta function concentration.) One could use impulses in the TF plane to represent a signal, however they are not feasible. Gabor proposed to use time and frequency shifted Gauss windows as basis functions because of their optimal TF concentration [3]. Hence the Gabor expansion represents a signal as a combination of time and frequency translated basis functions, called TF atoms. This type of basis functions generate a fixed and rectangular TF plane sampling. However, if the signal to be represented is not modeled well by this constant-bandwidth analysis, its Gabor representation displays poor TF localization [4,5,6,7]. Many of the real-world signals such as speech, music, machine vibrations, biological, and seismic signals however, have time-varying frequency content that is not appropriate for sinusoidal analysis [6,7,8]. Thus the Gabor expansion of such signals will require large number of Gabor coefficients yielding a poor TF localization. The compactness of the Gabor representation is improved if the basis functions match the time-varying frequency behavior of the signal [9,10]. Several approaches have been proposed to improve the resolution of Gabor representations: some of them are using large dictionary of basis functions [6,9], averaging results obtained using different windows [8], maximizing energy concentration measures [4,7,10], and using signal-adaptive basis functions to match the time-varying frequency of the signal [11]. In recent works, representations on a non-rectangular TF grid have attracted a considerable attention [10,12,13]. A non-rectangular lattice is more appropriate for the TF analysis of signals with time-varying frequency content. Here we present a discrete fractional Gabor expansion on such a sampling scheme. The basis functions of the proposed expansion are obtained from the closed-form discrete fractional Fourier transform [14] kernel.

2 Background

In this section we give a brief introduction to the discrete Gabor expansion and fractional Fourier analysis.

2.1 Discrete Gabor Expansion

The traditional Gabor expansion [3,15] represents a signal in terms of time and frequency shifted basis functions, and has been used in various applications to analyze the time-varying frequency content of a signal [8]. The discrete Gabor expansion of a finite-support signal $x(n)$, $0 \le n \le N-1$ is given by [15]

$$x(n) = \sum_{m=0}^{M-1}\sum_{k=0}^{K-1} a_{m,k}\tilde{h}_{m,k}(n) \qquad 0 \le n \le N-1 \qquad (1)$$

where the basis functions are

$$\tilde{h}_{m,k}(n) = \tilde{h}(n-mL)e^{jw_k n} \qquad (2)$$

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and $\omega_k = \frac{2\pi L'}{N}k$. The synthesis window $\tilde{h}(n)$ $h(n)$ is a periodic extension (by N) of $h(n)$ which is normalized to unit energy for definiteness [15]. The Gabor expansion parameters *M, K, L,* and L' are positive integers constrained by *ML=KL'=N* where *M* and *K* are the number of analysis samples in time and frequency, respectively, and *L* and L' are the time and frequency steps, respectively. For numerically stable representations, L and L' must satisfy $LL' \le N$, or equivalently that *L≤K* The case where *L=K*, is called the critical sampling, and the case *L<K* is called the over-sampling.

The Gabor coefficients $a_{m,k}$ can be evaluated by using an auxiliary function $\gamma(n)$ called the bi-orthogonal window or analysis function [16]:

$$a_{m,k} = \left\langle x(n), \tilde{\gamma}_{m,k}(n) \right\rangle = \sum_{n=0}^{N-1} x(n)\tilde{\gamma}^*_{m,k}(n) \qquad (3)$$

where again $\tilde{\gamma}(n)$ is a periodic version of the analysis window $\gamma(n)$ that is solved from a discrete biorthogonality condition between the analysis and synthesis bases sets [15].

Above Gabor basis $\tilde{\gamma}_{m,k}(n)$ with a fixed window and sinusoidal modulation tiles the TF plane in a rectangular fashion. Here we propose a discrete fractional Gabor expansion on a non-rectangular TF sampling lattice, using the discrete fractional Fourier transform kernel [14].

In [17] authors presented a continuous-time fractional Gabor expansion using basis functions similar to the kernel the fractional Fourier transform (FRFT) [18]. FRFT provides a rotation of the TF plane by an angle α. Quite amount of efforts have been made to define a discrete-time version of the FRFT [14,19].

2.2 Closed-form Discrete Fractional Fourier Transform

In [14], a closed-form discrete FRFT is given and ,t was shown that the kernel of this transform is orthogonal, unitary, and invertible. Basically, it is given by sampling the original kernel of the FRFT. However, the sampling is done in a way that the signal to be transformed and the final transform are sampled using the sampling parameters. Hence the transformation matrix obtained in this way is orthogonal. The input signal $f(t)$ and the transform $F_\alpha(u)$ are sampled by sampling intervals Δt and Δu *n=, -N,-N+1, . . ., N* and *m= -M, M+1, . . ., M* as,

$$y(n) = f(n.\Delta t), \quad Y_\alpha(m) = F_\alpha(m.\Delta u) \qquad (4)$$

Thus, the discrete FRFT of *y(n)* is defined as [14],

$$Y_\alpha(m) = \sqrt{\frac{1 - j\cot\alpha}{2\pi}}.\Delta t.e^{\frac{j}{2}.\cot\alpha.m^2\Delta u^2} \times \sum_{n=-N}^{N} e^{-j.\csc\alpha.n.m.\Delta u.\Delta t}.e^{\frac{j}{2}.\cot\alpha.n^2\Delta t^2}.y(n) \qquad (5)$$

or using the transformation kernel $F_\alpha(m,n)$,

$$Y_\alpha(m) = \sum_{n=-N}^{N} F_\alpha(m,n).y(n),$$

$$F_\alpha(m.n) = \sqrt{\frac{1 - j\cot\alpha}{2\pi}}..\Delta t.e^{\frac{j}{2}.\cot\alpha.m^2\Delta u^2} \times e^{-j.\csc\alpha.n.m.\Delta u.\Delta t}.e^{\frac{j}{2}.\cot\alpha.n^2\Delta t^2} \qquad (6)$$

The inverse transform is given for $M \ge N$ by

$$y(n) = \sum_{n=-N}^{N} F^*_\alpha(m,n).Y_\alpha(m), \quad M \ge N \qquad (7)$$

Under the condition that $\Delta u.\Delta t = S.2\pi.\sin\alpha/(2M+1)$ we obtain two definitions for the discrete FRFT for $\sin(\alpha)$>0 and $\sin(\alpha)$<0 cases:

$$Y_\alpha(m) = \sqrt{\frac{\sin\alpha - j\cos\alpha}{2M+1}}.e^{\frac{j}{2}.\cot\alpha.m^2\Delta u^2} \times \sum_{n=-N}^{N} e^{-j\frac{2\pi.n.m}{2M+1}}.e^{\frac{j}{2}.\cot\alpha.n^2\Delta t^2}.y(n), \quad \sin\alpha > 0 \qquad (8)$$

$$Y_\alpha(m) = \sqrt{\frac{-\sin\alpha + j\cos\alpha}{2M+1}}.e^{\frac{j}{2}.\cot\alpha.m^2\Delta u^2} \times \sum_{n=-N}^{N} e^{j\frac{2\pi.n.m}{2M+1}}.e^{\frac{j}{2}.\cot\alpha.n^2\Delta t^2}.y(n), \quad \sin\alpha < 0 \qquad (9)$$

For *M=N* and $\alpha = \pi/2$ the above definiton reduces to DFT and for α = -π/2 it becomes the inverse DFT.

3 Closed-form Discrete Gabor Expansion

A fractional Gabor expansion with fraction order α can be defined for a discrete-time signal *x(n) n=0, 1, ...N-1* (N odd) as follows:

$$x(n) = \sum_{m=0}^{M-1} \sum_{k=0}^{k-1} a_{mk\alpha} \tilde{h}_{mk\alpha}(n) \qquad (10)$$

The basis functions are

$$\tilde{h}_{mk\alpha}(n) = \tilde{h}(n - mL) K_\alpha(n,k) \qquad (11)$$

Here $\tilde{h}(n)$ is periodic version of the unit energy synthesis window $h(n)$:

$$\tilde{h}(n) = \sum_r h(n + rN)$$

M is the number of samples in time and K is the number of samples in the α fractional domain (that is mixed time and frequency domain u); L and L' are the samples in time and fractional u domain respectively, and the previous candition still holds:

$$ML = KL' = N$$

Furthermore $K_\alpha(n,k)$ is the fractional kernel and it replaces the kernel $\{e^{jw_k n}\}$ of the traditional Gabor expansion where the term $e^{jw_k n}$ modulates $h(n)$ synthesis window and shifts it in the frequency domain by L'. Similarly, in the fractional case, the kernel $K_\alpha(n,k)$ will shift $h(n)$ in the u domain by the same steps. A kernel $K_\alpha(n,k)$ which will provide such a shift can be obtained from the kernel of the closed-form discrete FRFT given above:

$$K_\alpha(n,k) = \sqrt{\frac{\sin\alpha - j\cos\alpha}{N}}\, e^{\frac{j}{2}\left[n^2\Delta t^2 + (kL')^2\Delta u^2\right]\cot\alpha}\, e^{-\frac{j2\pi nkL'}{N}}$$

subject to the constraint,

$\Delta u \Delta t = \dfrac{2\pi|\sin\alpha|}{N}$. The fractional Gabor coefficients $a_{mk\alpha}$ will be calculated as before by an analysis window $\gamma(n)$ that is biorthogonal to $h(n)$. Again the analysis window $\gamma(n)$ is periodized to obtain $\tilde{\gamma}(n)$ by

$$\tilde{\gamma}(n) = \sum_r \gamma(n + rN)$$

Then the Gabor coefficients are calculated as,

$$a_{mk\alpha} = \sum_{n=-(N-1)/2}^{(N-1)/2} x(n)\tilde{\gamma}^*_{mk\alpha}(n) \qquad (12)$$

where the analysis basis functions $\tilde{\gamma}_{mk\alpha}(n)$ are obtained by

$$\tilde{\gamma}_{mk\alpha}(n) = \tilde{\gamma}_{mk\alpha}(n - mL) K_\alpha(n,k) \qquad (13)$$

The completeness condition of this basis system is obtained by substituting $a_{mk\alpha}$ into the expanison in given (10):

$$x(n) = \sum_{l=-(N-1)/2}^{(N-1)/2} x(l) \underbrace{\sum_{m=0}\sum_{k=0} \underbrace{\tilde{h}_{mk\alpha}(n)\tilde{\gamma}^*_{mk\alpha}(l)}_{\delta(n-l)}}$$

and the condition is simplified to

$$\frac{1}{N}\sum_m \sum_k \tilde{h}(n-mL)\tilde{\gamma}^*(l-mL) \times e^{\frac{j}{2}\left[n^2-l^2\right]\Delta t^2\cot\alpha}\, e^{-\frac{j2\pi nkL'}{N}(n-l)} = \delta(n-l) \qquad (14)$$

The discrete fractional biorthogonality condition can be derived from the above completeness condition by Poisson sum formula as:

$$\frac{1}{LL'} \sum_n \left(\tilde{h}(n)\tilde{\gamma}^*(n - mR) e^{-j\frac{2\pi nk}{N}} \right) = \delta_m \delta_k \qquad (15)$$

where $R = -\dfrac{N}{L'}$ and for

$$0 \le k \le L-1,\ 0 \le m \le L'-1,\ -\frac{N-1}{2} \le n \le -\frac{N-1}{2}$$

For a given Gauss synthesis window, the analysis window $\gamma(n)$ can be solved from the above equation system and used to calculate the fractional Gabor coefficients.

4 Examples

Example 1. In this example we analyze a signal composed of two crossing chirps with angles $\pi/4$ and $-\pi/8$. Fig. 1 shows the traditional magnitude squared Gabor coefficients $|a_{m,k}|^2$. Notice that both components are represented with a poor localization. Then we analyzed this signal using the proposed discrete fractional Gabor expansion with fraction order $\alpha_1 = \pi/4$, and the resulting fractional Gabor coefficients $|a_{mk\alpha}|^2$ is given in Fig. 2. Now, the component of the signal that is matched by the analysis fraction order is represented in the TF plane with a higher localization than the other one. Finally, the signal is also analyzed with another fraction order $\alpha_2 = -\pi/8$ and the resulting $|a_{mk\alpha}|^2$ is shown

in Fig. 3. As seen in the figures, the component that is matched by the analysis angle is better represented in the TF plane than the others. Therefore, in order to represent all components of an arbitrary signal with an acceptable TF resolution, we need a search algorithm for optimal analysis angle. This can be achieved by analyzing the signal under investigation with a set of pre-determined fraction orders $\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ and the Gabor coefficients can be chosen by maximizing a concentration criteria similar to the method used in [4,10].

Example 2. To show this search method, we take a signal with quadratic instantaneous frequency (IF) and first analyzed using the traditional Gabor expansion. Fig. 7 shows the traditional Gabor TF spectrum $|a_{m,k}|^2$. Then we analyzed this signal using the proposed fractional method using angles in the interval $\alpha = [0 - \pi]$ with $\pi/36$ rad. increments. Then an adaptive search algorithm is employed to find the fractional Gabor coefficients with the highest possible TF localization [4] in small TF regions. The final fractional Gabor spectrum $|a_{mk\alpha}|^2$ is shown in Fig. 8. The contour plots of the traditional and the fractional Gabor spectra are given in Figs. 9 and 10.

5 Conclusions

In this paper, we present a method for fractional Gabor expansion for discrete-time, non-stationary signals. We give the completeness and biorthogonality conditions of this new expansion. Biorthogonal analysis function can be calculated for a chosen synthesis window using the fractional biorthogonality condition. Simulations show that the fractional method gives high resolution Gabor spectra if the analysis fraction order match the frequency component of the signal. Hence, for an arbitrary signal, the fraction order α can be chosen from a set of values $\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ by maximizing a concentration criteria similar to the method used in [4,10].

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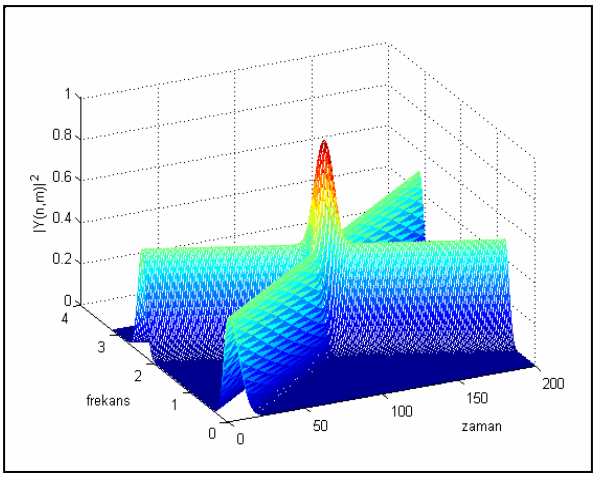

Figure 1. The sinusoidal Gabor coefficients of the crossing-chirp signal.

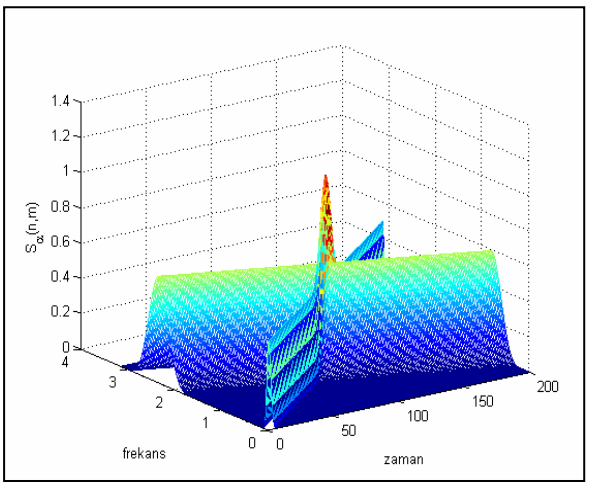

Figure 2. The fractional Gabor coefficients of the crossing-chirp signal with fraction order $\alpha = \pi/4$.

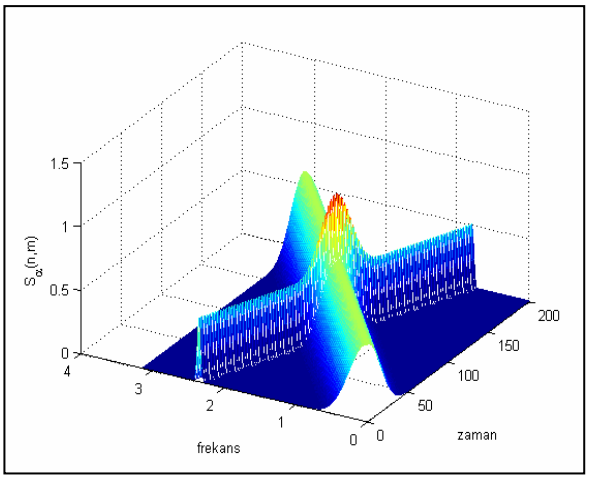

Figure 3. The fractional Gabor coefficients of the signal with fraction order $\alpha = -\pi/8$.

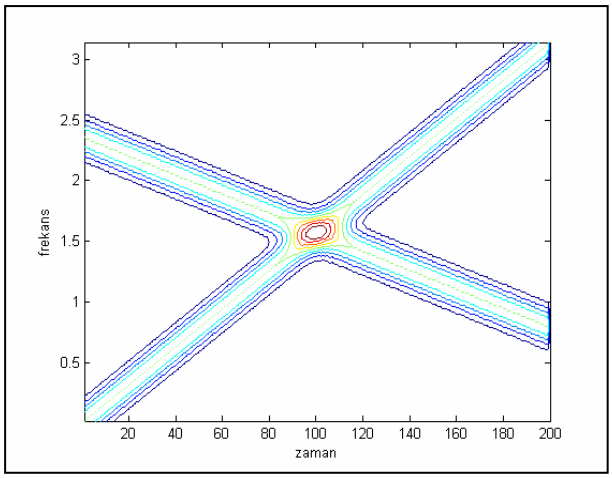

Figure 4. Contour plot of the sinusoidal Gabor coefficients of this signal.

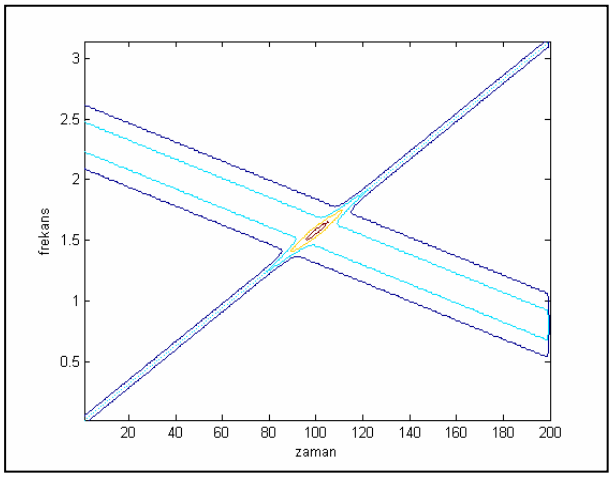

Figure 5. Contour plot of the fractional Gabor coefficients of this signal with fraction order $\alpha = \pi/4$.

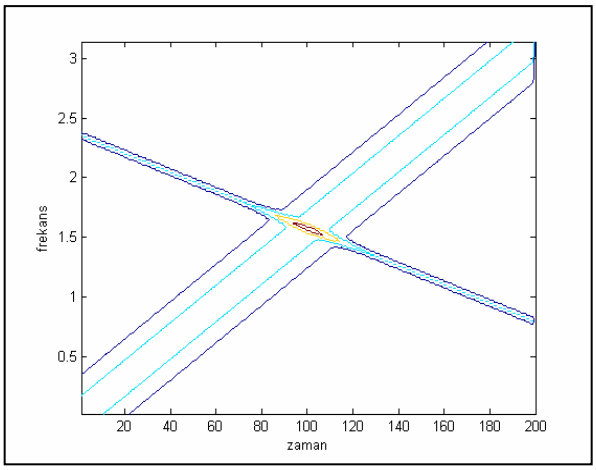

Figure 6. Contour plot of the fractional Gabor coefficients of this signal with fraction order $\alpha = -\pi/8$.

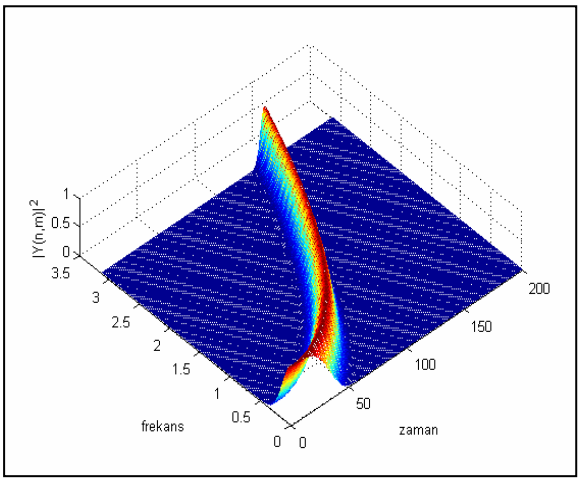

Figure 7. The traditional sinusoidal Gabor spectrum of the quadratic chirp signal.

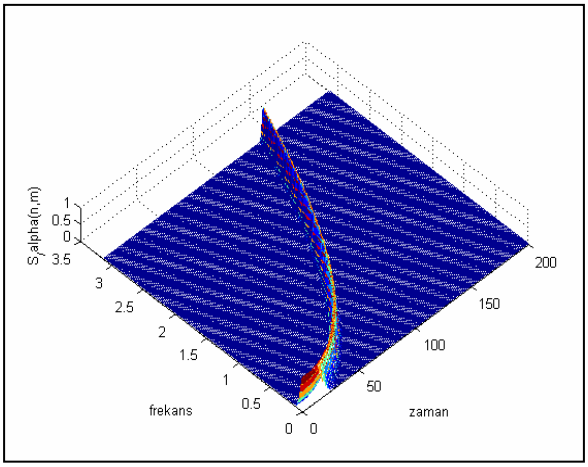

Figure 8. The combined fractional Gabor coefficients of the quadratic chirp signal using a search algorithm with $\alpha = \pi/36$ rad. increments.

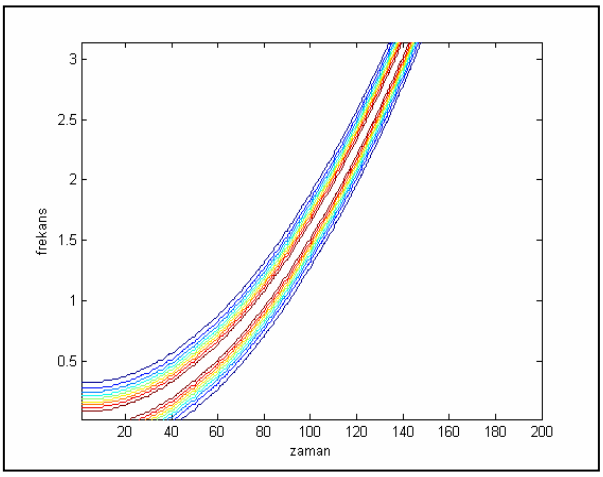

Figure 9. Contour plot of the traditional sinusoidal Gabor spectrum of the quadratic chirp signal.

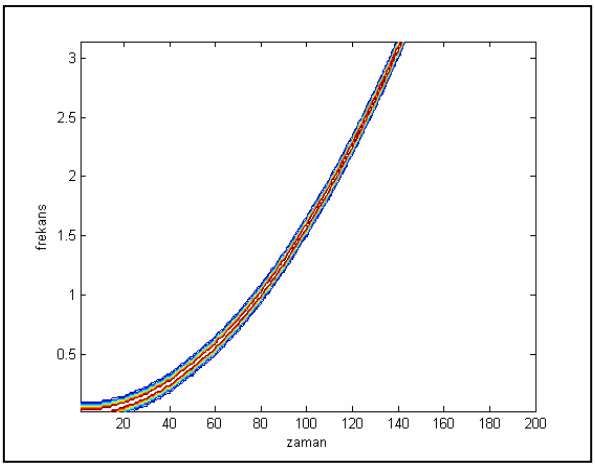

Figure 10. Contour plot of the combined fractional Gabor spectrum of this signal using a search algorithm with $\alpha = \pi/36$ rad. increments.