# ESTIMATION OF THE PERMANENT OF A BINARY MATRIX

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## Problem description

Let $A$ be a square matrix over an arbitrary field. The *permanent* of the matrix $A$ is defined as the algebraic sum of the products of any $N$ elements of the matrix, one in each row and column.

Symbol: *perm A* – the permanent of the matrix $A$.

Obviosly, the permanent is similar to the determinant, but without the sign change.

From now on only numeric matrices will be considered.

The evaluation of the permanent is a difficult mathematical problem. It has been put forward by Binet and Cauchy almost two centuries ago (cf. [1]). The fastest exact algorithm for the general case is Ryser's one, but it runs in $O(N.2^N)$ time. It has been proved that this problem is *#P*-hard, i.e. probably unsolvable in polynomial time. Then researchers have concentrated their efforts on discovering polynomial randomized algorithms. The latest results in this field can be found in [2], [3] and [4].

Many problems of practical importance can be reduced to the evaluation of the permanent. It is often the case that the efficiency of the algorithm is more signifficant than its precision.

Consider the so called rook problem: Given a binary matrix $N \text{ x } N$. A *density* of the matrix is defined as the count of the ones, divided by the count of all elements. A *density of a row/column* of the matrix is the ratio of the count of the ones in it to the count of its elements. An *assignment of length k* is defined as a set of k ones of the matrix, one in each row and each column. When $1 \le k < N$, the assignment is *partial*; when $k = N$, the assignment is *full*.

- Determine whether there exists an assignment.
- Find the count of assignments.
- Generate at least one full assignment.

Algorithms for the existence subproblem can be found in [5], [6] and [7], and for generating - in [8]. Here we shall discuss the subproblem for the count of the full assignments.

This problem can be treated as a special case of the famous assignment problem.

The next theorem gives the relationship between the permanent and the count of the assignments.

**Theorem 1 (about the permanent and the full assignments):** *Let A be a binary square matrix. Then the count of the full assignments in A is equal to perm A.*

Proof: *perm A* is a sum of ones and zeros; every one is formed as a product of ones, one in each column and each row. So every one corresponds to a full assignment and vice versa. Now the result immediately follows. ■

## Approximate formulae for the permanent

**Теорема 2**: *In a binary matrix $N \times N$ of a density $\rho$ the count of the full assignments is approximately equal to $N!\rho^N$.*

Proof: Let $Cnt(N,\rho)$ be the count of the full assignments in the matrix. This number is the sum of the count of the full assignments which pass through any of the ones in the last column (the sum is taken over all such ones). Through a fixed one there pass as many full assignments as their count in its adjunctive matrix. Since this matrix is of type $(N–1) \times (N–1)$, it differs "little" from the given matrix ($2N–1$ are eliminated which is one order less than the count of all elements), i.e. its density is approximately equal to $\rho$. Consequently, the count of the full assignments in the submatrix is approximately $Cnt(N–1,\rho)$. Hence $Cnt(N,\rho)$ is approximately equal to the count of the ones in the last column multiplied by $Cnt(N–1,\rho)$. And the count of the ones in the last column is approximately $N\rho$. So the following approximate equality holds:

$$Cnt(N,\rho) \approx (N.\rho).Cnt(N-1,\rho)$$

Applying this formula to the righthand side, we get:

$$Cnt(N,\rho) \approx (N.\rho).Cnt(N-1,\rho) \approx (N.\rho).((N-1).\rho).Cnt(N-2,\rho) \approx \ldots \approx$$
$$(N.\rho).((N-1).\rho). \ldots (3.\rho).(2.\rho).Cnt(1,\rho) \approx$$
$$(N.\rho).((N-1).\rho). \ldots (3.\rho).(2.\rho).(1.\rho) = N!\rho^N \quad \blacksquare$$

**Corollary 1**: *Let $A$ be a binary square matrix $N \times N$ of density $\rho$. Then $perm\,A \approx N!\rho^N$.*

**Theorem 3:** *Given an arbitrary binary matrix $N \times N$ whose rows/columns are of densities $\rho_1, \rho_2, \ldots, \rho_N$, the count of the full assignments in it is approximately $N!\,\rho_1.\rho_2 \ldots \rho_N$.*

Proof: Let $Cnt(N; \rho_1, \rho_2, \ldots, \rho_N)$ be the count of the full assignments. The next recurrent formula is obtained just like in the proof of theorem 2:

$$Cnt(N; \rho_1, \rho_2, \ldots, \rho_N) \approx (N.\rho_N).Cnt(N-1; \rho_1, \rho_2, \ldots, \rho_{N-1})$$

Applying it to the righthand side we get the result. $\blacksquare$

**Corollary 2:** *Let $A$ be a binary matrix $N \times N$ and its rows/columns be of densities $\rho_1, \rho_2, \ldots, \rho_N$. Then $perm\,A \approx N!\rho_1.\rho_2 \ldots \rho_N$.*

## An upper and a lower bound of the permanent

Another interesting problem is finding an upper and a lower bound of the permanent as functions of some easily computed charcteristics of the matrix. One can find them by using the ideas discussed in [5].

Consider the columns of the given matrix as sets. Namely, let $\Omega = \{1,2,\ldots,N\}$, $A = (a_{ij})$ be the matrix from the rook problem, $C_j$ be the set representing the $j$-th column of the matrix, $C_j = \{i \in \Omega \mid a_{ij} = 1\}$ for each $j \in \Omega$.

**Theorem 4 (F. Hall):** *There exists a full assignment in the matrix $A \Leftrightarrow$ for each $\Theta \subseteq \Omega$ the inequality $|\bigcup_{j \in \Theta} C_j| \geq |\Theta|$ holds. Moreover, if equality holds for* *some $\Theta$, the problem is decomposed in this sense: let $A_1$ be the submatrix of A, consisting of those columns whose indices belong to the set $\Theta$ and those rows whose indices are elements of the union; let $A_2$ be the submatrix of A consisting of the rest of the rows and columns; then each full assignment in A contains ones from the submatrices $A_1$ and $A_2$ only.*

Let $A$ be a binary matrix. A *null rectangle* is defined as any submatrix of $A$ all of whose elements are zeroes. A *goal* of a rectangle is defined as the sum of its width and height.

The name “goal” is justified by the fact that the algorithm in [5] searches for a null rectangle of a given goal.

**Corollary 3:** *Given a binary matrix A of type $N \times N$, there exists a full assignment $\Leftrightarrow$ there does not exist in A a null rectangle of goal greater than N. Moreover, if there exists a null rectangle $P \times Q$ of goal N, i.e. $P+Q = N$, then this rectangle decomposes the problem in this sense: let $A_1$ be the submatrix $P \times P$ of A consisting of the rows which enter the rectangle and the columns which do not enter it; let $A_2$ be the submatrix $Q \times Q$ of A consisting of the rows which do not enter the rectangle and the columns which enter it; then every full assignment in A contains ones from $A_1$ and $A_2$ only.*

Without loss of generality suppose that there exists a full assignment in the matrix. According to theorem 4, for each $\Theta \subseteq \Omega$ the inequality $|\bigcup_{j\in\Theta} C_j| \geq |\Theta|$ holds. Let $M$ be the set of all non-negative integers $k$ such that the inequality $|\bigcup_{j\in\Theta} C_j| \geq |\Theta| + k$ holds for each non-empty $\Theta \subseteq \Omega$ with $|\Theta| \leq N-k$ . Obviosly, $M$ is non-empty (it contains at least the number 0), it has a finite upper bound (for instance, $N-1$) and it contains all non-negative integers less than any element of its. So $M = \{0, 1, 2, ... , K\}$, where $K$ is the greatest element of $M$. This integer $K$ is called a *superiority* of the columns of the binary matrix.

On the one hand, there is a fast algorithm for finding the superiority; on the other hand, this quantity can be used for estimating the count of the assignments, and hence of the permanent.

**Theorem 5:** *Let A be a binary matrix N x N.*

*a) Let k be a whole integer, $0 \leq k < N$. Then the inequality $|\bigcup_{j\in\Theta} C_j| \geq |\Theta| + k$ holds for each non-empty $\Theta \subseteq \Omega$ with $|\Theta| \leq N-k \Leftrightarrow$ there does not exist in A a null rectangle of goal $> N-k$.*

*b) Let K be the superiority of the columns of A and G be the maximal goal of null rectangles in A (1, if no such rectangle exists). Then $K = N-G$.*

Proof:

a) ‘$\Rightarrow$’: First, let us make a remark – the theorem says nothing about those $\Theta \subseteq \Omega$ with $|\Theta| > N-k$. Since we have $|\bigcup_{j\in\Theta} C_j| = N$ when $|\Theta| = N-k$, it follows that $|\bigcup_{j\in\Theta} C_j| = N$ when $|\Theta| > N-k$ as well.

Suppose, on the contrary, that there is in $A$ a null rectangle $P \times Q$ of goal $P+Q > N-k$. Let $\Theta$ be the set of the indices of the columns of this rectangle.

Then $|\Theta| = Q$, $|\bigcup_{j\in\Theta} C_j| \le N-P < Q+k$. When $Q \le N-k$, the second inequality contradicts the condition of the theorem. If $Q > N-k$, then it follows from the first inequality that $|\bigcup_{j\in\Theta} C_j| \le N-P < N$, which contradicts the remark from the previous paragraph.

'⇐': Suppose, on the contrary, that there is a non-empty $\Theta \subseteq \Omega$ with $|\Theta| \le N-k$, such that $|\bigcup_{j\in\Theta} C_j| < |\Theta| + k$. Let $Q = |\Theta|$, $P = |\Omega \setminus \bigcup_{j\in\Theta} C_j|$, so $P > N-(Q+k) = N-k-Q \ge 0$. Columns whose indices belong to $\Theta$ and rows whise indices do not belong to $\bigcup_{j\in\Theta} C_j$ form a null rectangle $P$ x $Q$ of goal $P+Q > N-k$ what is a contradiction.

b) Case I: $K < N-1$

Then the inequality $|\bigcup_{j\in\Theta} C_j| \ge |\Theta| + k$ holds for each non-empty $\Theta \subseteq \Omega$ with $|\Theta| \le N-k$, where k = K, and this is not true for k = K+1. It follows from 'a' that there does not exist a null rectangle of goal $> N-K$, but there exists a one of goal $> N-(K+1)$. So $G = N-K$.

Case II: $K = N-1$

When $|\Theta| = 1$, it follows that every column has $N$ ones. Consequently, the matrix $A$ does not have zeroes, i.e. $G = 1 = N-K$.

So, in both cases $K = N-G$, q.e.d. ■

The algorithm from [5] which searches for a null rectangle of a given goal can be modified so that to search for $G$ (start with the smallest goal; each time the goal is reached increment it by one and continue searching for a null rectangle of the new goal; stop when the current search fails or when the maximal possible goal is reached; $G$ = the greatest reached goal or 1 if no goal is reached). Using theorem 5 we can find $K$.

**Theorem 6 (supremum of the count of the full assignments):** *Let A be a binary matrix N x N.*

*a) Let A contain a null rectangle P x Q, P+Q ≤ N–k. Then the count of the full assignments in A is* $\le \frac{(N-P)!(N-Q)!}{(N-P-Q)!}$. *Equality is reached ⇔ all elements of A out of this rectangle are ones.*

*b) Let K be the superiority of the columns of A. Then the count of the full assignments in A is ≤ (N–1)!(K+1). Equality is reached ⇔*

*1) K = N–1 or*

*2) K < N–1, the height or width of the null rectangle of maximal goal is equal to 1 and all elements of A out of this rectangle are ones.*

Proof:

a) The remark about the case when equality is reached, is obvious. From here we find the upper bound: consider a matrix which contains a null rectangle $P$ x $Q$, $P+Q \le N-k$, all element out of it are ones. Without loss of generality assume that its rows have indices 1, 2, ... , $P$; and its columns – 1, 2, ... , $Q$. One can take $Q$ rows from the last $N–P$ in $\binom{N-P}{Q}$ different ways. Together with the first $Q$ columns they form a submatrix $Q$ x $Q$, containling ones only. The count of the full assignments in this submatrix (they are partial assignments of length $Q$ for the matrix) is equal to $Q!$ . The rest $N–Q$ rows and columns also form a submatrix containing ones only; the count of the full assignments in this submatrix (they are partial assignments of length $N–Q$ for the matrix) is equal to $(N–Q)!$ . Combining the assignments of both submatrices, we obtain a full assignment in the matrix. Since every assignment in $A$ this way exactly once, it follows that the count of the full assignments in the matrix is equal to $\binom{N-P}{Q}.Q!\ (N-Q)! = \frac{(N-P)!(N-Q)!}{(N-P-Q)!}$

b) Let $G$ be the maximal goal of a null rectangle. Then $K = N–G$. If $G = 1$, i.e. $K = N–1$, then all elements of $A$ are ones, so the count of the full assignments is equal to $N!$ , i.e. equality is reached. Let $G > 1$, i.e. $K < N–1$. Then there exists in $A$ a null rectangle $P$ x $Q$ with $P+Q = G$. It follows from 'a' that the count of the full assignments in $A$ is $\leq \max_{P+Q=G} \frac{(N-P)!(N-Q)!}{(N-P-Q)!} = \frac{1}{(N-G)!} \cdot \max_{P+Q=G} (N-P)!(N-Q)!$ Since $(N–P)+(N–Q) = 2N–G$ is fixed, the maximum is reached when one of the integers $N–P$ и $N–Q$ is as big as possible, i.e. when $P = 1$ or $Q = 1$; this maximum is equal to $\frac{1}{(N-G)!}.(N–1)!(N–G+1)! = (N–1)!(K+1)$ ■

**Corollary 4 (supremum of the permanent):** *Let A be a binary matrix of type N x N.*

*a) Let there exist in A a null rectangle P x Q of goal P+Q ≤ N–k. Then perm A ≤* $\frac{(N-P)!(N-Q)!}{(N-P-Q)!}$*. Equality is reached ⇔ all elements of A out of this rectangle are ones.*

*b) Let K be the superiority of the columns of A. Then the following inequality holds: perm A ≤ (N–1)!(K+1). Equality is reached ⇔*

1) *K = N–1 or*
2) *K < N–1, the height or width of the null rectangle of maximal goal is equal to 1 and all elements of A out of this null rectangle are ones.*

We do not know a formula about the infimum (as a function of $N$ and $K$), neither the matrices it is reached in. However, it is possible to deduce an approximate estimate, which is enough for practice in almost all cases.

We say that a square binary matrix $A = (a_{ij})$ is *k-diagonal* (where $k \leq N–1$ is a non-negative integer), if $a_{ij} = 1$ ⇔ the remainder of $i – j$ modulo $N$ is some of the numbers 0, 1, 2, ... , $k$.

One property of the $k$-diagonal matrices is that they contain a lot of null rectangles, i.e. the hypothesis that the infimum is reached for these matrices, is probably true. This stimulates their deeper investigation.

**Theorem 7:** *The superiority of the columns of a k-diagonal matrix is k.*

Proof: For each non-empty $\Theta \subseteq \Omega$ with $|\Theta| \le N-k$ holds the inequality $|\bigcup_{j\in\Theta} C_j| \ge |\Theta| + k$. Moreover, for some $\Theta \subseteq \Omega$ (for example, $|\Theta| = 1$) equality is reached. So in the last inequality the integer $k$ can be substituted with a greater number. It follows the superiority of the columns is exactly $k$. ■

Since $k$-diagonal matrices have a "good" structure, we hope it is easy to deduce an exact formula for the count of the full assignments in them. If the hypothesis stated above is true, then this will be formula for the infimum of the count of the full assignments as a function of the size and the superiority of the columns of the matrix.

The first values of the count of the full assignments in a $k$-diagonal matrix $N$ x $N$ are given below:

| k / N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | |
| 2 | 1 | 2 | | | | | | |
| 3 | 1 | 2 | 6 | | | | | |
| 4 | 1 | 2 | 9 | 24 | | | | |
| 5 | 1 | 2 | 13 | 44 | 120 | | | |
| 6 | 1 | 2 | 20 | 80 | 265 | 720 | | |
| 7 | 1 | 2 | 31 | 144 | 579 | 1854 | 5040 | |
| 8 | 1 | 2 | 49 | 264 | 1265 | 4738 | 14833 | 40320 |

**fig. 1**

To our regret, there is known no exact formula for the count of the full assignments in a $k$-diagonal matrix, as shows the check-up in the Encyclopedia of Integer Sequences [9] (the identification number of the sequence is A008305), except a one that represents the counts of the full assignments as a permanent of a matrix. But we are interested in reducing the problem for the permanent to the problem for the count of assignments.

In [9] one can find formulae for some special cases. Obviously, for $k = N-1$ we get the sequence of factorials. For $k = N-2$ we get the sequence of subfactorials, which give the count of the so called derangements (permutations with no fixed point). There are formulae for $k = N-3$ and $k = N-4$.

**Theorem 8:** *Let Cnt be the count of the full assignments in a k-diagonal matrix of type N x N.*

*a) If $k = N-1$, then $Cnt = N!$*

*b) If $k = N-2$, then* $Cnt = N!\sum_{j=0}^{N}\frac{(-1)^j}{j!} = \left[\frac{N!}{e}+\frac{1}{2}\right]$

*c) If $k = N-3$, then* $Cnt = \sum_{j=0}^{N}\frac{(-1)^j.2N.\binom{2N-j}{j}.(N-j)!}{2N-j}$

We adduce these formulae for exposition fullness only.

However, it is easy to find an approximate formula for the general case.

**Theorem 9 (infimum of the count of the full assignments):** *The infimum of the count of the full assignments in binary matrices N x N of a superiority of columns K, as well as the count of the full assignments in K-diagonal matrices of order N are approximately equal to* $N!\left(\frac{K+1}{N}\right)^N \approx N!\, e^{K+1-N}$

Proof: Let $A$ be a binary matrix of type $N \times N$ and of superiority of columns $K$. So every column contains at least $K$+1 ones, i.e. the density of each column is no less than $\frac{K+1}{N}$. It follows from theorem 2 that the count of the full assignments in $A$ is gretaer than or equal to $N!\left(\frac{K+1}{N}\right)^N$. On the other hand, in a $K$-diagonal matrix of type $N \times N$ every column contains exactly $K$+1 ones, i.e. the density of each column is exactly $\frac{K+1}{N}$, so the count of full assignments in such a matrix is approximately equal to $N!\left(\frac{K+1}{N}\right)^N$. Consequently, the number $N!\left(\frac{K+1}{N}\right)^N \approx N!\, e^{K+1-N}$ is approximately equal to the infimum and to the count of the full assignments in a $K$-diagonal matrix of type $N \times N$. ■

**Corollary 5 (infimum of the permanent):** *The infimum of the permanent of binary matrices N x N of a superiority of columns K, as well as the permanent of K-diagonal matrices of order N are approximately equal to* $N!\left(\frac{K+1}{N}\right)^N \approx N!\, e^{K+1-N}$

Since the equalities from theorem 3 and corollary 2 are approximate, so are the equalities in theorem 9 and corollary 5; the only thing they show is that in the $k$-diagonal matrix $N \times N$ the infimum is approximately reached. However, in the general case an exact equality does not hold; see this counterexample:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

**fig. 2**

This matrix is of type 6 x 6, the superiority of its columns is 2, its permanent is equal to 18. But the permanent of the 2-diagonal matrix 6 x 6 is equal to 20.

Nevertheless, $k$-diagonal matrices are useful, because they are close to the infimum.

## References


[1] Hentyk Minc, Permanents, Encyclopedia of Mathematics and its Applications 6 (1982), Addison-Wesley Publishing Company

[2] Mark Jerrum, Alistair Sinclair, Eric Vigoda, A polynomial-time approximation algorithm for the permanent of a matrix with non-negative entries, Electronic Colloquium on Computational Complexity, Report № 79 (2000), pp. 1149–1178, http://citeseer.nj.nec.com/532431.html

[3] Alex Sivkins,Approximate Counting, CS 683:Advanced Algorithms,2001, www.cs.cornell.edu/Courses/cs683/2001SP/lec17.ps

[4] Avi Widgerson, Computational Complexity Theory, 1999, http://www.cs.huji.ac.il/~amirs/complex/ex/ex4.ps

[5] Dimov D., Kralchev D., Penev A., Stanchev S., Existence of solutions to the assignment problem, International conference in automatics and informatics, Sofia, May 30 – June 2, 2001, pp. I-81– I-83

[6] Kralchev D., Dimov D., Penev A., An algebraic method for solving the rook problem, Plovdiv University “Paissii Hilendarski”, Bulgaria, Scientific works, vol. 33, book 3, 2001 - Mathematics, pp. 53–60 стр.

[7] Kralchev D., Dimov D., Penev A., Statistical analysis of the rook problem, “Mathematical forum”, vol. 4, book 4, Sofia, 2002 (in Bulgarian)

[8] Kralchev D., Dimov D., Penev A., Generating an assignment in the rook problem, “Mathematical forum”, vol. 5, book 1, Sofia, 2003 (in Bulgarian)

[9] Sloane N.J.A., On-Line Encyclopedia of Integer Sequences, http://www.research.att.com/~njas/sequences/Seis.html


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