Abstract—A time signal distribution system is an important part of several engineering assemblies being present in process controlling equipment, distributed computation systems and telecommunication networks. In order to have accurate performance, synchronous operation of these systems, composed by several nodes, needs a reliable time basis signal extracted from the line data stream in each node. When the nodes are synchronized, routing and detection can be performed, guaranteeing the correct sequence of information distribution among the several users of a shared circuit. Consequently, an auxiliary network is created inside the main circuit, a sub-network, dedicated to the distribution of the clock signals.

There are different solutions for the architecture of the time distribution sub-network and choosing one of them depends on cost, precision, reliability and operational security.

In this work an overview of the possible time distribution schemes is given. Additionally, a detailed study of a robust option is presented by using the qualitative theory of differential equations. Correspondences between constitutive parameters of the network and the dynamics of the spatial phase and frequency errors are established.

Keywords: Bifurcation, Dynamical Systems, equilibrium, master-slave network, phase-locked loop, synchronous network

I. INTRODUCTION

The analysis of geographically separated oscillators started to become an important problem for telecommunications in the sixties with the introduction of the first digital trunks which required synchronous timing basis for demodulation and regeneration of pulse code modulation (PCM) signals [9], [11].

The phase-locked loop is a device introduced by Belescize [3] in 1932 to extract timing signals. Nowadays, it is used in integrated circuit versions with high precision and low cost [4]. This device can extract the clock from digital signals corrupted by distortion and noise in transmission media. They were initially used in regenerators and termination units of digital multiplexing equipment [15], [14].

Nowadays, in telecommunications, this kind of circuits are useful in higher hierarchy multiplexing systems, in order to guarantee synchronization among several digital streams of lower hierarchy in the terminal stations [2], [19].

In processing control systems they can provide the synchronization among measuring, control and sensor systems guaranteeing the correct execution of supervising and control tasks, in real time.

Parallel distributed architectures and computational clusters need synchronized clock signals, provided by PLLs, in order to establish the peer to peer connections between system layers.

In this work, the time distribution system that supports these applications is analyzed considering that the several possible solutions are based in the dynamics of the phase-locked loop.

The idea is to show that, in spite of the problem complexity, using Dynamical System theory is an interesting tool in order to obtain robustness, controllability and reachability conditions for synchronous states.

II. PHASE-LOCKING PROBLEM

The problem of phase-locking consists of controlling the phase of a local oscillator by the phase of an external oscillator, making them coincide or, at least, differ by a constant.

From the point of view of electronic engineering, a phase-locked loop is the device that accomplishes it. It is a closed loop system connecting three basic elements: a phase detector (PD), a filter (F) and a voltage-controlled oscillator (VCO) [4], [8]. A basic PLL is shown in figure 1.

Fig. 1. Block diagram of a PLL

The input and output signals are, respectively, given by:

\[ v_i(t) = V_i \sin(\omega_0 t + \theta_i(t)), \]
\[ v_o(t) = V_0 \cos(\omega_0 t + \theta_0(t)) \]

In these expressions, \( \omega_0 \) is the central frequency here named free-running frequency of the loop, \( \theta_i(t) \) and \( \theta_0(t) \) are the instantaneous phases, and \( V_i \) and \( V_0 \) are the amplitudes of \( v_i(t) \) and \( v_o(t) \).

The loop is considered to be in a locked or synchronous state when it reaches an equilibrium state, with constant phase error \( \phi = \theta_i - \theta_0 \) and null frequency error \( \dot{\phi} = \dot{\theta}_i - \dot{\theta}_0 \) [4], [8].

As the phase detector is a signal multiplier, the PD output is given by:

\[ v_d(t) = \frac{1}{2} K_m V_i V_0 \left[ \sin(\theta_i - \theta_0) + \sin(2\omega_0 t + \theta_i + \theta_0) \right], \]

where \( K_m \) is the phase detector gain.
The filter is supposed to eliminate high frequency terms. So, if the double frequency term is sufficiently attenuated by the filter \(F\), equation (1) is reduced to:

\[ v_d(t) = K_d \sin(\theta_t - \theta_0), \]  

(2)

with \(K_d = \frac{1}{2}K_m V_i V_o\), in volts per radian.

In many real engineering situations, the linear approximation of the sine by its argument is considered, reducing the PD output to:

\[ v_d(t) = K_d(\theta_t - \theta_0). \]

Here the model of PD and its output \(v_d(t)\) follow equation (2). Being simple, the filter \(F\) is an all-pole low-pass with zeros in infinite [16], [5] with transfer function:

\[ F(s) = \frac{V_e(s)}{V_d(s)} = \frac{b_0}{s^n + b_{n-1}s^{n-1} + \cdots + b_0}, \]

(3)

where \(V_e(s)\) and \(V_d(s)\) represent the Laplace transforms of signals \(v_e(t)\) and \(v_d(t)\), respectively.

The combination of equations (2) and (3) yields:

\[ \frac{d^n}{dt^n} v_e(t) + b_{n-1} \frac{d^{n-1}}{dt^{n-1}} v_e(t) + \cdots + b_0 v_e(t) = b_0 K_d \sin(\theta_t - \theta_0). \]

(4)

The output phase \((\theta_0)\) of the VCO is controlled by \(v_e(t)\) and satisfies \(\theta_0 = K_0 v_e\), where \(K_0\) is a VCO constant, in radians per volt per second [8]. Thus, equation (4) can be rewritten as:

\[ \frac{d^{n+1}}{dt^{n+1}} \theta_0(t) + b_{n-1} \frac{d^n}{dt^n} \theta_0(t) + \cdots + b_0 \frac{d}{dt} \theta_0(t) = b_0 K_0 K_d \sin(\theta_t - \theta_0). \]

(5)

Defining \(L(\cdot)\)

\[ L(\cdot) = \frac{d^{n+1}}{dt^{n+1}}(\cdot) + b_{n-1} \frac{d^n}{dt^n} (\cdot) + \cdots + b_0 \frac{d}{dt} (\cdot). \]

and by taking the phase error \(\varphi(t) = \theta_t - \theta_0\) as the dynamic variable, equation (5) becomes:

\[ L(\varphi) + b_0 K_0 K_d \sin(\theta_t - \theta_0) = L(\theta_t). \]

(6)

The ordinary differential equation (6) describes the behaviour of a phase-locked loop that is the main component of circuits for extracting time signals.

**III. DISTRIBUTION OF TIMING SIGNALS**

The problem of time distribution along networks consists of controlling frequency and phase of clock signals spreading over a wide area. The idea is synchronizing the frequency and phase scales of several oscillators in a network by using the data communication capacity of the links.

This problem has several applications [13]:

- Establishing a world wide time distribution system.
- Synchronizing clocks located at different points in a digital communication network.
- Distributing time signals in a network in order to apply control actions and commands at specific times.
- Establishing a supercomputer by interconnecting several computers in a network.

These items are sufficient to justify the relevance of timing distribution in applications related to control and communication engineering.

In real problems, objective comparisons among the several possibilities are needed. Then, a precise mathematical treatment is necessary.

As it was already stated, the intention is to discuss the several strategies for spreading clock signals and the synchronization of several oscillators distributed over a wide geographic area.

There are situations in which precision in synchronization is not a critical point. In these cases, independent clocks manually adjusted are used. This strategy originated the plesiochronous networks.

When synchronization results from interactions between the oscillators of the network, it is called synchronous.

Synchronous networks with a clock priority mechanism are called master-slave. When all the clocks in a network have equal relevance in determining the synchronous state, it is called mutually synchronized.

In what follows, the phases of local oscillators, denoted by \(\Phi\), are composed by a free-running term \(\omega t\), a forcing term \(\theta(t)\) and a perturbation \(P(t)\), i.e.,

\[ \Phi(t) = \omega t + \theta(t) + P(t). \]

Master-slave networks are classified according to the transmission direction of time basis in One-Way Master-Slave (OWMS) and Two-Way Master-Slave (TWMS).

In OWMS networks, the master clock has its own and independent time basis. Slave clocks have their basis depending on another node, the master or another slave. Besides, these networks are classified according to the topology in chain and star.

In TWMS networks, the master clock has its own time basis but the control signal sent to the slave clocks is adjusted according to the basis of other nodes. Slave clocks may have their time basis dependent on several nodes.

According to the topology, TWMS networks can be classified as chain, star or loop.

Master-slave networks are extensively adopted in public telecommunication networks due to simple implementation, good timing performance, reliability, and low cost [6]. They also have applications in parallel distributed computation [20], robotics [12], and multimedia applications [21].

In this work the TWMS double-star network is studied. As figure 2 shows, the master, called node 1, has an accurate and independent time basis. However the control signal that it sends to the slaves considers its own phase and the phase of all slaves.

Figure 3 shows a scheme of master nodes in TWMS networks, indicating the mechanism for generating control signals considering the phase of the master \(\Phi_M\) and the phase of the slaves \(\Phi_i\).

Control signals \(\Phi\) sent by the master to the network is submitted to a weighting process that considers all the phase of the slaves with coefficients \(a_{i,j}\) such that \(\sum_{j=2}^{N} a_{1,j} = 1\).
Fig. 2. Double-star TWMS network

In this work, related to the double-star scheme, all the slaves are considered with the same relevance, consequently $a_{1,j} = 1/(N-1)$, $\forall j = 2, \ldots, N$.

In a TWMS network, PLLs belonging to the slave nodes have an input signal with phase $\Phi$ resulting from a linear combination of phases from the several nodes, as shown in figure 4. The linear combination follows the condition $\sum_{j=2}^{N}a_{i,j} = 1$ in each $i$th-slave.

In double-star systems any slave-node do not contribute to the input signal of the other slaves. Therefore: $a_{i,1} = 1$ and $a_{i,j} = 0$, $\forall i, j = 2, \ldots, N$.

IV. STABILITY OF EQUILIBRIUM STATES AND FREQUENCY ERRORS IN A DOUBLE-STAR TWMS NETWORK

When a TWMS strategy of clock distribution is chosen one has a more robust and accurate performance for the network. But, in this situation, due to the feedback loops between the nodes, frequency errors, even low, propagate along the whole network spoiling the performance.

In this section, the problem of frequency error propagation in double-star TWMS networks is studied by using techniques from dynamical systems theory [18] obtaining conditions of keeping then controllable [10].

The slaves considered are second order PLLs with a time constant $\mu$. The architecture is the double-star with a master $M$ and $N-1$ slaves.

The master is an oscillator with phase $\phi_M(t)$. Signal propagation time from the master to the $i$th slave is indicated by $\tau_{1i}$, and, from the $i$th slave to the master, by $\tau_{i1}$, for $i = 2, \ldots, N$.

Phases of oscillators output in this network are defined as follows:

- Master oscillator
  \[
  \phi_M(t) = \omega_M t + P_M(t). 
  \]

- $i$th slave-node oscillator
  \[
  \phi_i(t) = \theta_i(t) + \omega_i t + P_i(t), \quad i = 2, 3, 4, \ldots, N. 
  \]

- Master output phase
  \[
  \phi_1(t) = 2\phi_M(t) - \frac{1}{N-1} \sum_{i=2}^{N} \phi_i(t - \tau_{i1}). 
  \]

Modeling each $i$th slave, $i = 2, 3, 4, \ldots, N$, with a PLL equation, as we have seen in section II, their dynamics can be described as follows:

\[
\ddot{\phi}_i(t) + \mu \dot{\phi}_i(t) - \mu \mu_i \sin(\phi(t - \tau_{1i}) - \phi_2(t)) = \dot{P}_i(t) + \mu \omega_i + \mu \dot{P}_i(t), \tag{10}
\]

where $\mu$ is the filter cut-off frequency in all nodes, and $\mu_i$ is the $i$th slave-node PLL gain.

Defining frequency and phase spatial errors by:

\[
\begin{align*}
\varphi_{M,i} &= \phi_M - \phi_i, \\
\varphi_{M,i} &= \phi_M - \phi_i. 
\end{align*} 
\tag{11}
\]

The substitution of equations (7), (8) and (9) in (10), taking into account equation (11), results:

\[
\begin{align*}
\ddot{\varphi}_{M,i} + \mu \varphi_{M,i} + \mu \mu_i \sin(\frac{N}{N-1} \varphi_{M,i} + \frac{1}{N-1} \sum_{j=2}^{N} \varphi_{M,j}) & = \frac{1}{N-1} \sum_{j=2}^{N} (\tau_{1i} + \tau_{j1}) \varphi_{M,j} - \frac{1}{N-1} (\omega_M + \Omega_M t) \\
& + \frac{1}{N-1} \sum_{j=2}^{N} \tau_{j1} (P_M(t) + \mu \dot{P}_i(t) \\
& + \mu \omega_i + \Omega_M t) + \mu \dot{P}_i(t) \\
& - \mu \dot{P}_i(t). 
\end{align*} 
\tag{12}
\]

The obtained dynamics is non-linear and depends explicitly on time, so there is no equilibrium state. The oscillator degradation combined with the delays does not allow the system to be locked in the steady state [7], [17].

If the derivatives in equation (12) are taken and considering a linear approximation by expanding the non-linear terms in
Taylor series [8], [7], [17], one can write:

\[ \ddot{\varphi}_{Mi} + \mu \dot{\varphi}_{Mi} + \mu \mu_i \left[ \frac{N}{N-1} \ddot{\varphi}_{Mi} + \frac{1}{N-1} \sum_{j=2}^{N} N \right] \]

\[ - \frac{1}{N-1} \sum_{j=2}^{N} (\tau_{ii} + \tau_{ij}) \dot{\varphi}_{Mj} - \frac{1}{N-1} \left[ (N-1) \tau_{ii} \right] + \sum_{j=2}^{N} \tau_{jj} \Omega_M = \mu (\Omega_M - \Omega_i) \tag{13} \]

Considering state variables:

\[ x_{2i-3} = \dot{\varphi}_{Mi}, \quad \text{and} \quad x_{2i-2} = \ddot{\varphi}_{Mi}, \]

the system becomes:

\[
\begin{align*}
\dot{x}_{2i-3} &= x_{2i-2}, \\
\dot{x}_{2i-2} &= -\mu x_{2i-2} - \mu \mu_i \left[ \frac{N}{N-1} \sum_{j=2}^{N} x_{2j-3} \right] \\
&\quad - \frac{1}{N-1} \left[ (N-1) \tau_{ii} - \sum_{j=2}^{N} \tau_{jj} \Omega_M \right] + \mu (\Omega_M - \Omega_i).
\end{align*}
\]

This system admits an equilibrium state which corresponds to constant frequency spatial errors \( \dot{\varphi}_{Mi} \) and non-limited phase spatial errors \( \ddot{\varphi}_{Mi} \).

Acceleration spatial error \( \ddot{\varphi}_{Mi} \) tends to a zero stationary state. After the transient states, the acceleration of any slave follows the acceleration of the master.

The linear part of the new system, around the equilibrium state, can be represented by:

\[
\begin{bmatrix}
0 & \mu \mu_2 \tau_{22} + \tau_{21} & 0 & \cdots & 0 \\
\mu \mu_2 \tau_{22} + \tau_{21} & \ddots & \ddots & \ddots & \ddots \\
0 & \mu \mu_2 \tau_{22} + \tau_{21} & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \mu \mu_N \tau_{N2} + \tau_{N1} & 0 \\
\end{bmatrix}
- \mu \mu_2 \tau_{22} + \tau_{21} \frac{N}{N-1} \sum_{j=2}^{N} \tau_{jj} \Omega_M
\]

\[
\begin{bmatrix}
0 & \mu \mu_2 \tau_{22} + \tau_{21} & 0 & \cdots & 0 \\
\mu \mu_2 \tau_{22} + \tau_{21} & \ddots & \ddots & \ddots & \ddots \\
0 & \mu \mu_2 \tau_{22} + \tau_{21} & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \mu \mu_N \tau_{N2} + \tau_{N1} & 0 \\
\end{bmatrix}
- \mu \mu_2 \tau_{22} + \tau_{21} \frac{N}{N-1} \sum_{j=2}^{N} \tau_{jj} \Omega_M
\]

Calculating the eigenvalues of \( A \) by using MAPLE V [1]:

\[ \lambda_1 = \lambda_2 = \ldots = \lambda_r = -\frac{\mu}{2} + \sqrt{\mu^2 - 4\mu \nu}, \]

\[ \lambda_{r+1} = \lambda_{r+2} = \ldots = \lambda_{2(N-2)} = -\frac{\mu}{2} - \sqrt{\mu^2 - 4\mu \nu}, \]

\[ \lambda_{2(N-1)-1,2(N-1)} = \mu \nu - \frac{\mu}{2} \pm \sqrt{(\mu \nu - \frac{\mu}{2})^2 - 2\mu \nu}. \]

Examining the eigenvalues one can conclude that the equilibrium point of the system, having constant acceleration error, is asymptotically stable for any physical possible value of the parameters.

V. CONCLUSION

As a result of long-term instability of the master oscillator in a double star TWMS network, slave nodes do not synchronize in phase with master. Phase spatial error \( \dot{\varphi}_{Mi} \) between a slave and the master is unlimited as a consequence of equation (12).

However, in some practical situations, the propagation times can be related to the gains of the PLLs making the frequency errors controllable.

Summarizing:

- It follows from equation (12) that the double-star TWMS network does not present synchronous solution when oscillators suffer a phase acceleration and frequency errors propagate along the whole network.
It follows from (16) that the frequency spatial errors do not depend on the number of slaves but only on the acceleration of the master and the slaves.

Restrictions on the stability domain of equilibrium state depend on the relation between PLLs gains and signal delays.

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