The polynomial power filter of the second degree for filtering first-type skewness signal in Gaussian interference

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Abstract: - In this paper the optimum noncausal polynomial power filter of the second degree is synthesized for filtering of the useful nonGaussian signal of an additive mixture of the signal and the Gaussian interference. The useful signal concerns the first type of skewness random processes. It is shown that in this case the optimum polynomial power filter consists of three linear filters, namely of the usual optimal linear filter, the complementary filter of the first degree and the optimal linear filter for the square of the input signal (linear filter of the second degree). In the case where the third order cumulant function of the useful signal is equal to zero, then the complementary filter of the first degree and optimum linear filter of the second degree are equal to zero and the polynomial degree filter changes into the usual optimum linear filter.

Key-Words: - non-linear filter, nonGaussian signal, skewness random process, cumulants, spectral equations.

1 Introduction

Linear filters have found broad application in technical systems of different purposes due to their simplicity and the minimum of a priori information which is necessary for the synthesis of optimal linear filters. In fact, in the synthesis of optimal linear filters the correlation functions of the useful signal and interference are only used which describe Gaussian random processes. Recently, papers has appeared in which as useful signals are used nonGaussian random processes. In this connection there is a necessity for the construction of filters which in their synthesis would take into account the nonGaussian useful signal. These filters are non-linear filters. There are different approaches to the construction of non-linear filters. In [1] the polynomial power filters are presented, which are non-linear filters, but as with linear filters they are easy to implement and, at the same time, they allow us to take into account and to effectively use the nonGaussian useful signal in description in the form of sequences of cumulant functions of the higher orders of a two-dimensional distribution.

The purpose of this paper is the synthesis of the polynomial power filter of the second degree for the filtering of nonGaussian useful signals, namely the first type of skewness random process. The same random signals are described only by the correlation function and the third order cumulant function.

2 Statement of the problem

Let the observable random signal $\xi(t)$ represent an additive mixture of the useful signal $s(t)$ and interference $n(t)$. We will consider that the signal and interference are the real and fourth order stationary random processes independent of each other with expectations equal to zero. Let the signal be the nonGaussian stochastic process relating to the first type of skewness random process which is described by the correlation function $\chi_2(\tau)$ and the third-order cumulant function $\chi_3(\tau)$ of two-dimensional distributions [2,3].
Let the interference be the Gaussian random process with the correlation function \( \eta_2(\tau) \).

The task is to synthesize the optimum polynomial power filter of the second degree to filter of the useful non-Gaussian signal \( s(t) \) which is in the class of the first type of skewness random processes.

### 3 Synthesis of the optimum polynomial power filter of the second degree

The most widespread non-linear filters are the filters based on the use of the Volterra series [4]. However these filters difficult to implement technically and to investigate mathematically. Polynomial power filters are simpler and are thus used in this paper.

#### 3.1 The polynomial power filter

According to the theory of polynomial power filters [1], as the estimation of the useful signal \( \hat{s}_2(t) \) the noncausal polynomial power filter of the second degree is used. It has the form

\[
\hat{s}_2(t) = h_0 + \int_{-\infty}^{\infty} h_1(\tau) \xi(t-\tau)d\tau + \int_{-\infty}^{\infty} h_2(\tau) \xi^2(t-\tau)d\tau. \tag{1}
\]

where \( h_1(\tau) \) and \( h_2(\tau) \) are pulse responses of linear filters for the observable signal in the first and second degree respectively. Therefore the polynomial power filter of the second degree in the common case consists of the additive combination of two linear filters. Thus the observable signal \( \xi(t) \) acts on the input of one linear filter and the square of this signal acts on the input of the other linear filter. Therefore the polynomial power filter of the second degree is the non-linear filter in relation to an input signal \( \xi(t) \).

In [1] it were emphases that the first requirement which is presented to the filter is the requirement of a unbiasedness of the estimation of the useful signal. For the polynomial power filter of the form (1) the estimation will be unbiased if the constant \( h_0 \) is equal to

\[
h_0 = -m_1 \int_{-\infty}^{\infty} h_1(\tau) d\tau - m_2 \int_{-\infty}^{\infty} h_2(\tau) d\tau. \tag{2}
\]

For this constant the estimation of the useful signal has the form

\[
\hat{s}_{(c)2}(t) = \int_{-\infty}^{\infty} h_1(\tau) \xi(t-\tau) - m_1 d\tau + \int_{-\infty}^{\infty} h_2(\tau) \xi^2(t-\tau) - m_2 d\tau. \tag{3}
\]

The second requirement of the polynomial power filter is the requirement that the pulse responses of linear filters of the first and second degree provide a minimum of mean-square error of a filtering. It is possible to show, that the expression for the mean-square error of filtering of the skewed useful signal of the first type by the polynomial power filter of the second degree is equal to

\[
C_2[s(t) ; \hat{s}_c(t)] = \chi_2 - 2 \sum_{i=1}^{2} \int_{-\infty}^{\infty} h_1(\tau) R_{1,i}(\tau) d\tau + \sum_{i=1}^{2} \sum_{j=1}^{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\tau) h_1(\nu) F_{ij}(\tau-\nu) d\tau d\nu, \tag{4}
\]

where correlanten functions as \( R \)-type are equal to

\[
R_{1,1}(\tau) = \chi_2(\tau), \quad R_{1,2}(\tau) = \chi_1(\tau),
\]

and correlanten functions as \( F \)-type have the form:

\[
F_{1,1}(\tau-\nu) = \chi_2(\tau-\nu) + \eta_2(\tau-\nu),
F_{1,2}(\tau-\nu) = \chi_1(\tau-\nu),
F_{2,2}(\tau-\nu) = 2[\chi_2(\tau-\nu) + \eta_2(\tau-\nu)]^2.
\]

From expression (4) it can be seen that the mean-square error of the filtering is equal to the classical functional Volterra polynomial. In this functional correlanten functions of the \( R \) - and \( F \)-types are used as given kernels and the pulse responses of linear filters of the first and second degree are acted on as independent variables of functions. The mean-square error (4) is the two-dimensional functional parabola with respect to pulse responses \( h_1(\tau) \) and \( h_2(\tau) \). The functional polynomials of a
parabolic type have extremals. Therefore, the pulse responses $h_1(\tau)$ and $h_2(\tau)$ which ensure a minimum of the mean-square error are extremals of the expression (4).

3.2 Optimum pulse responses of the polynomial power filter

Applying the calculus of variations it is possible to obtain the result that the set of equations for finding extremals of the expression (4) is a system of integral equations [1] of the form

$$
\int_{-\infty}^{\infty} h_1(v)F_{1,1}(\tau-v)dv + \int_{-\infty}^{\infty} h_2(v)F_{1,2}(\tau-v)dv = R_{1,1}(\tau),
$$

where $R_{1,1}(\tau)$ is the transfer function of the degree $i$, $F_{ij}(\omega)$ and $R_{1,1}(\omega)$ are spectra of appropriate correlaten functions.

The solution of system (6) is transfer functions of linear filters of the first and second degree:

$$
\tilde{h}_1(\omega) = \frac{\tilde{\chi}_2(\omega)\tilde{F}_{1,2}(\omega) - \tilde{\chi}_3(\omega)}{\tilde{\Delta}_2(\omega)},
$$

$$
\tilde{h}_2(\omega) = \tilde{\chi}_3(\omega)\frac{\tilde{n}_R(\omega)}{\tilde{\Delta}_2(\omega)},
$$

where $\tilde{\Delta}_2(\omega)$ is the volume of the spectral body of size $2 \times 2$.

The expression for the transfer function $\tilde{h}_1(\omega)$ can be written in the more comprehensible form:

$$
\tilde{h}_1(\omega) = \frac{\tilde{\chi}_2(\omega) - \tilde{\chi}_3(\omega)}{\tilde{F}_{1,1}(\omega)\tilde{\Delta}_2(\omega)}. \quad (7)
$$

In essence, the new result in comparison with the linear filter is the dependence of both transfer functions of the spectrum of the third order cumulant function $\tilde{\chi}_3(\omega)$ on the useful signal. And as can be seen from (7), the transfer function $\tilde{h}_1(\omega)$ consists of an additive combination of two transfer functions. One transfer function is equal to the transfer function of a usual optimal linear filter and the second transfer function is directly proportional to the square of the spectrum of the cumulant function $\tilde{\chi}_3(\omega)$. In turn the transfer function $\tilde{h}_2(\omega)$ is directly proportional to a spectrum $\tilde{\chi}_3(\omega)$. Thus, if the spectrum $\tilde{\chi}_3(\omega)$ is equal to zero than the transfer function $\tilde{h}_2(\omega)$ is equal to zero and the transfer function $\tilde{h}_1(\omega)$ changes into the transfer function of the usual optimal linear filter.

Extremals of the functional polynomial (4) or optimum pulse responses of linear filters of the first and second degree of the polynomial power filter of the second degree (3) are as follows, corresponding to the pulse responses:

$$
h_1^*(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\chi}_2(\omega) e^{j\sigma\tau} d\omega - \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\chi}_3(\omega) \frac{\tilde{n}_R(\omega)}{\tilde{\Delta}_2(\omega)} e^{j\sigma\tau} d\omega,
$$

$$
h_2^*(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\chi}_3(\omega) \frac{\tilde{n}_R(\omega)}{\tilde{\Delta}_2(\omega)} e^{j\sigma\tau} d\omega. \quad (8)
$$

Definition 1. The filter of the form (1) in which coefficient $h_o$ has the form (2) and the pulse responses $h_1^*(\tau)$ and $h_2^*(\tau)$ of linear filters of the first and second degree are equal (8) is called the optimum polynomial power filter of the second degree. The signal obtained
at the output of the optimal polynomial power filter is called the optimal polynomial estimation of the useful signal.

From the first expression (8) it is obvious that the pulse response of the linear filter of the first degree is equal to the sum of two pulse responses. One response is equal to the pulse response of the usual optimum linear filter and the other pulse response emerges from created non-Gaussian useful signal. Therefore, the optimal linear filter of the first degree consists of the usual optimum linear filter and the filter the output signal of which seems to complement the output signal of the optimum linear filter.

**Definition 2.** A linear filter with pulse response

\[ h_{(c)}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\chi}_2^2(\omega) \frac{\hat{n}_L(\omega)}{F_{1,1}(\omega)\Delta_2(\omega)} e^{j\omega \tau} d\omega. \]

is called the optimum linear filter which complements the linear filter of the first degree or the complementary filter of the first degree.

4 Conclusions

If we analyze the optimal polynomial power filter of the second degree for filtering of the first type of skewness useful signal it is possible to draw the conclusion that it consists of an additive combination of three linear filters, namely the usual optimum linear filter, the complementary filter of the first degree and optimum linear filter of the second degree. Thus the square of an input signal acts on the input of the linear filter of the second degree. Therefore the optimum polynomial filter of the second degree is the non-linear filter concerning of the input signal.

References: