MRAC USING OBSERVERS WITH UNKNOWN INPUTS

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Abstract: In this paper we present a model reference adaptive control scheme for plants with unity relative degree. The novelty of this approach is that the model reference need not be strictly real positive and the method uses the theory of observers with unknown inputs. Even though the analysis starts very general, the same basic assumptions for the MRAC are encountered when proving stability of the proposed method, but from a different viewpoint. Simulation results are included to compare the new approach with the standard MRAC.

Keywords: Model reference adaptive control, state observers, observers with unknown inputs, adaptive control.

1. Introduction

Model Reference Adaptive Control (MRAC) is a pretty well known and established methodology, where four basic assumptions have to be satisfied for its solution (Narendra and Annaswamy, 1989). These assumptions are the following:

i) Order of the plant known (or an upper bound)
ii) Sign of the high frequency gain known
iii) Plant relative degree known
iv) Zeros of the plant in the left half of the complex plane.

Although some of these assumptions have been relaxed, the assumption on the plant zeros is the most restrictive.

In the case of plants with unity relative degree the assumption that the model reference has to be strictly positive real (SPR) is introduced. In the more general case of plants with arbitrary relative degree, the concept of augmented error has to be introduced.

Using as starting point the corresponding Error Model by Narendra and Annaswamy [12] we attempt to solve the MRAC problem following a different route to that using the SPR concept. The key concept used to solve the problem is the construction of an observer for the error model, which has an unknown input, since the ideal controller parameters are unknown. In this attempt we found along the way the same assumptions introduced in the standard solution except that the reference model need not to be SPR, although it has to be of unity relative degree. This is a surprising fact since from two different sides we get the same general conditions on the transfer functions of the reference model.

The observability of linear time-invariant systems with unknown inputs was first studied by Basile and Marro [1], right after the results of Luenberger on state observers [10]. Since then, several papers have been published on this subject. [3, 6, 7, 8, 9, 11, 16].

Even recently some papers have been published [4, 5, 13, 14] indicating the importance of this theme.

In designing the observer with unknown inputs we follow the method proposed by Wang, Davison and Dorato [15], although some of the other proposed methods cited in the references could have also been used, conducting to the same general results.

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In Section 2 we state the general problem of the MRAC and the corresponding Error Model equation is derived. In Section 3 a solution to this problem, based on the design of an observer with unknown input, is presented. Some simulation results for a second order plant, comparing the standard and proposed solutions are shown in Section 4. Finally, some conclusions are drawn in Section 5.

2. Problem Statement

The MRAC is a control technique in which the output of a plant \( y_p(t) \) is desired to follow the output of an asymptotically stable reference model \( y_m(t) \) driven by a bounded reference \( r(t) \). To this purpose a control structure like the one that follows is commonly used

\[
\begin{align*}
u(t) &= \theta^T(t) \omega(t) \\
w(t) &= \left[r(t), w_1^T(t), y_p(t), w_2^T(t)\right] \in \mathbb{R}^{2n} \\
\theta(t) &= \left[k(t), \theta_1^T(t), \theta_0(t), \theta_2^T(t)\right] \in \mathbb{R}^{2n}
\end{align*}
\]

where

\[
\begin{align*}
\dot{w}_1(t) &= A w_1(t) + \ell u(t), & w_1 &\in \mathbb{R}^{n-1} \\
\dot{w}_2(t) &= A w_2(t) + \ell y_p(t), & w_2 &\in \mathbb{R}^{n-1}
\end{align*}
\]

This whole structure (plant, controller and model reference) can be suitably condensed in a new structure called Error Model which make the analysis simpler \[12\]. In this case, the Error Model has the following form:

\[
e(t) = \frac{k_p}{k_m} W_m(s) \left[\phi^T(t) \omega(t)\right]
\]

where

\[
W_m(s) = \frac{Z(s)}{R(s)} = \frac{k_m}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}
\]

In the standard solution of the MRAC it is assumed that the transfer function \( W_m(s) \) is SPR, then the adaptive laws are readily derived as

\[
\dot{\phi}(t) = -\text{sgn}(k_p e(t)) \omega(t)
\]

Our problem is now to find adaptive laws for \( \theta(t) \) such that

\[
\lim_{t \to \infty} e(t) = 0
\]

without assuming that \( W_m(s) \) is SPR.

Here we propose an alternate solution which imply an observer design for a linear time-invariant system with unknown inputs.

3. Model Reference Controller

If we express the equation (1) in state variables we can write

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + b \phi^T(t) \omega(t) \\
e(t) &= \frac{k_p}{k_m} W_m(s) \phi^T(t) \omega(t)
\end{align*}
\]

(2)

Since \( W_m(s) \) is a known transfer function, matrices \( A, b \) and \( c \) are known and their structures can be chosen by the designer in any way he likes (e.g. controllable, observable, Jordan, etc.).

Let us first assume that the state vector \( x(t) \) is accessible. Then the adaptive law to be chosen is

\[
\dot{\phi}(t) = (x(t) P b) \omega(t)
\]

where \( P \) is the solution of the Lyapunov equation

\[
A^T P + PA = -Q
\]

where \( Q \) is an arbitrary positive definite matrix.

To prove stability it is enough to choose the Lyapunov Function

\[
V(x, \phi) = x^T(t) P x(t) + \phi^T(t) \phi(t)
\]

Resulting its time derivative

\[
\dot{V}(x, \phi) = -x^T(t) Q x(t) \leq 0
\]

Unfortunately the state \( x(t) \) is not known and it has to be estimated (observed). The additional difficulty with
respect to the standard Luenberger observer is the fact
the input to the system \((\phi^T(t)\omega(t))\) is unknown. To get
around this problem we use the results by Wang,
Davison and Dorato [15]. In this sense we built a state
observer for system (2) of the following form:

\[ x = \begin{bmatrix} E & F \end{bmatrix} \begin{bmatrix} e \\ \alpha \end{bmatrix} \]

equation (2) can be written as

\[
\begin{bmatrix} \dot{e} \\ \alpha \end{bmatrix} = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix} \begin{bmatrix} e \\ \alpha \end{bmatrix} + \begin{bmatrix} K_5 \\ K_6 \end{bmatrix} \begin{bmatrix} T \\ \phi \omega \end{bmatrix} + (K_6 - TK_5) \begin{bmatrix} T \\ \phi \omega \end{bmatrix}
\]

\[ e = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} e \\ \alpha \end{bmatrix} \]

The state observer is then defined by the following
expression:

\[
\dot{\hat{\beta}} = [K_4 - TK_2] \dot{\beta} + [K_3 + K_4 T - TK_1 - TK_2 T] e + (K_6 - TK_5) \begin{bmatrix} T \\ \phi \omega \end{bmatrix} + \begin{bmatrix} b_{n-2} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix}
\]

and the following conditions must be satisfied by the
matrices

i) \([K_4 - TK_2]\) stable matrix

ii) \((K_6 - TK_5) = 0\)

Writing \(W_m(s)\) in its Observer Canonical form:

\[
\begin{bmatrix} \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_{n-1} & 1 & 0 \\ -a_{n-2} & \ddots & \vdots \\ -a_0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} -b_{n-1} \\ -b_{n-2} \\ \vdots \\ -b_0 \end{bmatrix} \begin{bmatrix} T \\ \phi \omega \end{bmatrix}
\]

\[ e = \frac{k p}{k m} \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ \vdots \\ x_n \end{bmatrix} \]

Then we can identify the following vectors and
matrices

\[ K_1 = -a_{n-1}, \quad K_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n-1} \]

\[ K_3 = \begin{bmatrix} -a_{n-2} \\ \vdots \\ -a_1 \\ -a_0 \end{bmatrix} \in \mathbb{R}^{n-1} \]

\[ K_4 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \in (n-1) \times (n-1) \]

\[ K_5 = b_{n-1}, \quad K_6 = \begin{bmatrix} b_{n-2} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix} \in \mathbb{R}^{n-1} \]

\[ E = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & 1 \end{bmatrix} \]

With this observer design the adaptive laws have the form

\[ \dot{\phi} = -\text{sgn}(k_p) \begin{bmatrix} T \\ \phi \omega \end{bmatrix} \begin{bmatrix} P \cdot b \omega(t) \end{bmatrix} \]

where the state estimate is given by (3). The stability
proof uses a suitable Lyapunov function involving the
state estimate and the controller parameter error.

Surprisingly, imposing conditions i) and ii) of the
design observer results the following constraints on the
reference model transfer function:

- Zeros stable
- Relative degree one
- Asymptotically stable.

These three conditions not necessarily means that
\(W_m(s)\) be SPR and correspond to the classical MRAC
assumptions [12].
4. Simulation Results

In order to compare the proposed solution with the classical MRAC, we have chosen a second order plant of unity relative degree to be controlled with both methods. The plant transfer function is

\[ W_p(s) = \frac{s + 2}{s^2 - s - 2} \]

and the model reference transfer function chosen is

\[ W_m(s) = \frac{s + 7}{s^2 + 3s + 2} \]

Note that \( W_m(s) \) is not SPR in general since \( \text{Re}(W_m(jw)) > 0 \) if \( w < 1.85 \) rad/sec.

The observable canonical form chosen to represent \( W_m(s) \) was

\[
\begin{bmatrix}
-3 & 1 \\
-2 & 0
\end{bmatrix} \cdot x(t) + \begin{bmatrix} 1 \\
7
\end{bmatrix} \cdot r(t)
\]

\[ y_m = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot x \]

The observer design used for the proposed solution has the following form

\[
E = \begin{bmatrix} 1 \\
0
\end{bmatrix}, \quad F = \begin{bmatrix} 0 \\
1
\end{bmatrix}, \quad K_1 = -3, \quad K_2 = 1, \quad K_3 = -2, \quad K_4 = 0, \quad K_5 = 1, \quad K_6 = 7.
\]

\[
\dot{x} = \begin{bmatrix} e \\
\alpha \end{bmatrix} = \left( \begin{bmatrix} 1 \\
0 \end{bmatrix} + \begin{bmatrix} 0 \\
1 \end{bmatrix} \cdot t \right) \cdot e + \begin{bmatrix} 0 \\
1 \end{bmatrix} \cdot \beta
\]

\[
\dot{\beta} = -t \cdot \beta + \left( -2 + 3t - t^2 \right) \cdot e + (7 - t) \phi^T \omega
\]

**Conditions:**

i) \( 7 - t = 0 \)

ii) \(-t \) stable \( \Rightarrow \) \( t = 7 \)

then

\[
\dot{x} = \begin{bmatrix}
e
\alpha
\end{bmatrix} = \begin{bmatrix} 1 \\
7
\end{bmatrix} \cdot e + \begin{bmatrix} 0 \\
1
\end{bmatrix} \cdot \beta
\]

\[
\dot{\beta} = -7 \cdot \beta + -30 \cdot e
\]

Figures 1 and 2 show the performance of the standard MRAC for a constant reference input of amplitude 4.

Fig. 1. Standard MRAC. Step reference input

Fig. 2. Proposed MRAC. Step reference input.

Figures 3 and 4 show the simulation results for the case of sinusoidal reference input for which \( W_m(s) \) is not SPR (amplitude 4 and frequency 4 rad/sec.)
Fig. 3. Standard MRAC. Sinusoidal reference input.

Fig. 4. Proposed MRAC. Sinusoidal reference input.

Fig. 5. Standard MRAC. Sinusoidal reference input and SPR condition.

Fig. 6. Proposed MRAC. Sinusoidal reference input and SPR condition.

Figures 5 and 6 show the case where a sinusoidal reference input in the range where $W_m(s)$ is SPR was chosen (Amplitude 4 and frequency 0.1 rad/sec).

From these Figures and several other simulations done but not presented here for the sake of space, it can be concluded that both schemes work suitably when $W_m(s)$ is SPR. In the case when $W_m(s)$ is not SPR, standard MRAC is able to guarantee only convergence of the control error but not asymptotic convergence as in the proposed method. In the standard MRAC, when the reference input is chosen in a frequency range in which $W_m(s)$ can be considered SPR, asymptotic convergence is achieved. In the proposed method, asymptotic convergence of the control error is achieved for all type of reference inputs.
5. Conclusions

A new solution for MRAC, based on the design of a state observer with unknown inputs has been proposed. The method is applicable to unity relative degree plants and allows to relax the SPR condition on the model reference transfer function used in the standard solution of the MRAC. The remaining assumptions of classical MRAC are encountered from a different viewpoint, showing an interesting link between MRAC and state observers.

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