Abstract: In this paper the discrete Fourier transform (DFT) is applied for determining the transfer function coefficients for second-order generalized state space linear systems, $E(x^{(2)}) = A_1 x + A_2 + b u$. The proposed algorithm is theoretically attractive, practically fast and has been implemented in Matlab. A step by step example illustrating the application of the algorithm is given.

1 Introduction

The study of generalized linear systems has experienced a great deal of interest in recent years. First-order generalized systems are applied in engineering as well as in biological and economic systems [1]–[4]. Second-order generalized systems, applied to power systems, were studied by Campbell and Rose [5]. These systems were also used in conjunction with the analysis and modelling of flexible beams [6].

The state space approach has been an active area of research in system theory. This is because, using state space methods, complex systems can be studied efficiently using digital computers. To represent a system, described by a state space model, with a transfer function and vice versa is a very important factor in system theory and control systems analysis because various problems can be studied more efficiently [2], [3].

In this paper a new algorithm is presented for the computation of the transfer functions for generalized second-order systems using the discrete Fourier transform (DFT). The DFT has also been used for computing the transfer functions for regular, first-order generalized and multidimensional systems [7]–[10].

2 Background

A second-order generalized linear system is described by the following state space equations:

\[
\begin{align*}
E x^{(2)} &= A_1 x + A_2 + bu \\
y &= c' x
\end{align*}
\]

where, $x^{(2)}$ denotes the second derivative of $x$ with respect to time. $x \in \mathbb{R}^\lambda, u \in \mathbb{R}^p, y \in \mathbb{R}^m$, matrices $E, A_i, \forall i = 1, 2$ are real with dimensions $\lambda \times \lambda$, and matrix $E$ may be singular with rank $\mu$. Applying the Laplace transform to (1,2), with zero initial conditions, the corresponding transfer function is found to be

\[
T(s) = c'(s^2 E - s A_1 - A_2)^{-1} b \tag{2}
\]

where, regularity is assumed, $|s^2 E - s A_1 - A_2| \neq 0$.

In the following section an interpolative approach is developed for determining the transfer function $T(s)$, given the matrices $E, A_i, (\forall i = 1, 2)$, $b$ and $c'$, using the DFT. For the sake of completeness a brief description of the DFT follows.

Given the finite sequences $X(k)$ and $\tilde{X}(r), k = 0, \ldots, N$, the following relationships are necessary in order for the sequences to constitute a DFT pair [11]:

\[
\tilde{X}(r) = \sum_{k=0}^{N} X(k) W^{-kr} \tag{3}
\]

\[
X(k) = \frac{1}{(N + 1)} \sum_{r=0}^{N} \tilde{X}(r) W^{kr} \tag{4}
\]

where $r = 0, \ldots, N$, $k = 0, \ldots, N$, and

$X = [x_{ij}], \tilde{X} = [\tilde{x}_{ij}], \ i = 1, \ldots, p, \ and \ j = 1, \ldots, m$ with

$W = e^{(2\pi j)/(N + 1)}$.

In the following section an interpolative approach is developed for determining the transfer function $T(s)$, given the matrices $E, A_i, (\forall i = 1, 2)$, $B$ and $C$, using the DFT.
3 Main result - algorithm

Let the transfer function, \( T(s) \), of the second-order generalized system be defined as

\[
T(z, w) = \frac{N(s)}{d(s)}
\]

(5)

where,

\[
N(s) = c' \text{adj} [s^2E - sA_1 - A_2]b
\]

(6)

\[
d(s) = \text{det} [s^2E - sA_1 - A_2]
\]

(7)

Taking into consideration that the degree of the characteristic polynomial \( \text{deg}[N(s)] = \text{det} [s^2E - sA_1 - A_2] \leq 2r = r \), equations (6) and (7) can be written in polynomial form as follows:

\[
N(s) = \sum_{k=0}^{r} P_k s^k
\]

(8)

\[
d(s) = \sum_{k=0}^{r} q_k s^k
\]

(9)

where \( P_k \) are matrices with dimensions \((p \times m)\), while \( q_k \) are scalars.

The numerator polynomial matrix \( N(s) \) and the denominator polynomial \( d(s) \) can be numerically computed at \((r+1)\) points, equally spaced on the unit circle as,

\[
v(i) = W^{-i}, \quad \forall \; i = 0, \ldots, r.
\]

(10)

where,

\[
W = e^{(2\pi i)/(r+1)}
\]

(11)

The values of the transfer function (5) at the \((r+1)\) points form its corresponding DFT coefficients.

3.1 Denominator polynomial

To evaluate the denominator coefficients \( q_k \), define

\[
a_i = \text{det} [E v(i) - v(i)A_1 - A_2]
\]

(12)

Using equations (7) and (12), \( a_i \) can also be defined as

\[
a_i = d_i [v(i)]
\]

(13)

provided that at least one of \( a_i \neq 0 \).

Equations (9), (10) and (13) yield

\[
a_i = \sum_{k=0}^{r} q_k W^{-ik}
\]

(14)

In the above equation (14), \([a_i], [q_k]\) form a DFT pair. Therefore the coefficients \( q_k \) can be computed using the inverse 1-D DFT, as follows:

\[
q_k = \frac{1}{(r+1)} \sum_{i=0}^{r} a_i W^{ik}
\]

(15)

where \( k = 0, \ldots, r \).

3.2 Numerator polynomial

To evaluate the numerator matrix polynomial \( P_k \), define

\[
F_i = c' \text{adj} [v^2(i)E - v(i)A_1 - A_2]b
\]

(16)

provided that at least one of \( F_i \neq 0 \).

Using equations (6) and (16), \( F_i \) can also be defined as

\[
F_i = N[v(i)]
\]

(17)

Equations (8), (10) and (17) yield

\[
F_i = \sum_{k=0}^{r} P_k W^{-ik}
\]

(18)

In the above equation (18), \([F_i], [P_k]\) form a DFT pair. Therefore the coefficients \( P_k \) can be computed using the inverse 1-D DFT, as follows:

\[
P_k = \frac{1}{(r+1)} \sum_{i=0}^{r} F_i W^{ik}
\]

(19)

where, \( k = 0, \ldots, r \).

Finally, the transfer function sought is

\[
T(s) = \frac{N(s)}{d(s)} = \frac{\sum_{k=0}^{r} P_k s^k}{\sum_{k=0}^{r} q_k s^k}
\]

(20)

Two salient examples, simple yet illustrative of the theoretical concepts presented in this work, follow below:

4 Examples

4.1 First Example

Consider the following second-order generalized system

\[
E x^{(2)} = A_1 x + A_2 + Bu
\]

(21)

\[
y = Cx
\]

where,
\[ E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \]

\[ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \]

We would like to determine the transfer function for this system using the technique outlined above. Note that the system satisfies the regularity condition.

Since \( \text{rank}(E) = \mu = 1 \), therefore \( r = 2\mu = 2 \).

The direct application of the proposed algorithm, for \( r + 1 = 0,1,2,3 \) points, yields

\[ W = \exp(j2\pi/4) = j \]

and

\[
\begin{align*}
W^0 &= +1 \\
W^{-1} &= -j \\
W^{-2} &= -1 \\
W^{-3} &= +j
\end{align*}
\]

Using (12), yields

\[
\begin{align*}
a_0 &= +1 \\
a_1 &= -4 - j \\
a_2 &= -1 \\
a_3 &= -4 + j
\end{align*}
\]

From (16) follows

\[
\begin{align*}
F_0 &= 1 \\
F_1 &= 3 \\
F_2 &= 1 \\
F_3 &= 3
\end{align*}
\]

Using (15), the denominator coefficients are

\[
\begin{align*}
q_0 &= -2 \\
q_1 &= 1 \\
q_2 &= 2 \\
q_3 &= 0
\end{align*}
\]

Using (19), the numerator coefficients are

\[
\begin{align*}
F_0 &= 2 \\
F_1 &= 0 \\
F_2 &= -1 \\
F_3 &= 0
\end{align*}
\]

Once the denominator and the adjoint matrix have been computed, equation (20) can be utilized to obtain the transfer function \( T(s) \). Therefore, we obtain

\[ T(s) = \frac{P_2 s^2 + P_0}{q_2 s^2 + q_1 s + q_0} \]

or

\[ T(s) = \frac{-s^2 + 2}{2s^2 + s - 2} \]

The result can be easily verified using (2).

### 4.2 Second Example

Consider the following second-order generalized system

\[ Ex^{(2)} = A_1 x + A_2 + Bu \]

\[ y = Cx \]

where,

\[ E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \]

\[ B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \]

We would like to determine the transfer function for this system using the technique outlined above. Note that the system satisfies the regularity condition.

Since \( \text{rank}(E) = \mu = 1 \), therefore \( r = 2\mu = 2 \).

The direct application of the proposed algorithm, for \( r + 1 = 0,1,2,3 \) points, yields

\[ W = \exp(j2\pi/4) = j \]

and

\[
\begin{align*}
W^0 &= +1 \\
W^{-1} &= -j \\
W^{-2} &= -1 \\
W^{-3} &= +j
\end{align*}
\]

Using (12), yields

\[
\begin{align*}
a_0 &= +1 \\
a_1 &= -4 - j \\
a_2 &= -1 \\
a_3 &= -4 + j
\end{align*}
\]

From (16) follows

\[
\begin{align*}
F_0 &= 1 \\
F_1 &= 3 \\
F_2 &= 1 \\
F_3 &= 3
\end{align*}
\]

Using (15), the denominator coefficients are

\[
\begin{align*}
q_0 &= -2 \\
q_1 &= 1 \\
q_2 &= 2 \\
q_3 &= 0
\end{align*}
\]

Using (19), the numerator coefficients are

\[
\begin{align*}
F_0 &= 2 \\
F_1 &= 0 \\
F_2 &= -1 \\
F_3 &= 0
\end{align*}
\]

Once the denominator and the adjoint matrix have been computed, equation (20) can be utilized to obtain the transfer function \( T(s) \). Therefore, we obtain

\[ T(s) = \frac{P_2 s^2 + P_0}{q_2 s^2 + q_1 s + q_0} \]

or

\[ T(s) = \frac{-s^2 + 2}{2s^2 + s - 2} \]

The result can be easily verified using (2).
Using (19), the numerator coefficients are

\[
\begin{align*}
P_0 &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
P_1 &= \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \\
P_2 &= \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \\
P_3 &= 0
\end{align*}
\]

Once the denominator and the adjoint matrix have been computed, equation (18) can be utilized to obtain the transfer function \( T(s) \). Therefore, we obtain

\[
T(s) = \frac{P_2 s^2 + P_1 s + P_0}{q_1 s^2 + q_1 s + q_0}
\]  

or

\[
T(s) = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} s^2 + \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} s + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}
\]

or

\[
T(s) = \begin{bmatrix} -s & 2s^2 - s - 1 \\ s^2 & -s \end{bmatrix}
\]

The result can be easily verified using (2).

### 5 Conclusion

A new algorithm was presented for the computation of the transfer function for second order generalized systems. The technique is based on the DFT algorithm. To improve the computational speed of the proposed algorithm fast Fourier methods can be used. The results presented in this paper can be extended to n-th order generalized systems.

### 6 References


