**General Form of Optimised State Model of the Second-Order Dynamical System**

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Abstract: - The conditions for optimized design of the second dynamical system having low eigenvalue sensitivities are derived. Their more general form is accordance with the previous results obtained by using linear topological conjugacy. 

Key-Words: - Dynamical System, State Models, Eigenvalue Sensitivities, Optimised Model 

1 Introduction

Recently published new state models of piecewise-linear (PWL) dynamical systems of Class C can be used as prototypes for their circuit realization. For this purpose their eigenvalue sensitivities have been minimized by using linear topological conjugacy [1], first for linear systems and then applied to PWL systems. By detailed analysis of the second order systems the new optimization conditions giving minimum sum of relative eigenvalue sensitivity squares with respect to the change of the individual state matrix parameters have been derived [2]. Applying them separately to the cases of real and complex conjugate eigenvalues, two different forms of the optimized state matrix are obtained [3]. In this contribution the generalized form of directly derived optimization conditions is introduced. It evidently includes both previous forms [3] as two special cases. 

2 Basic principles

Consider second-order linear dynamical system described by general state matrix form 

$$\mathbf{x} = \mathbf{A} \mathbf{x}, \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \quad (1)$$ 

Its characteristic equation 

$$\det(s \mathbf{1} - \mathbf{A}) = s^2 - (a_{11} + a_{22})s + (a_{11}a_{22} - a_{12}a_{21}) = 0. \quad (2)$$ 

has the roots given by 

$$a_{11} + a_{22} = \lambda_1 + \lambda_2, \quad a_{11}a_{22} - a_{12}a_{21} = \lambda_1 \lambda_2. \quad (3a,b)$$ 

The relative eigenvalue sensitivities with respect to the individual state matrix parameters are 

$$S_r(\lambda_k, a_{ij}) = \frac{a_{ij}(\lambda_k - a_{22})}{\lambda_k [2\lambda_k - (a_{11} + a_{22})]}, \quad (4a)$$ 

$$S_r(\lambda_k, a_{11}) = \frac{a_{11}(\lambda_k - a_{22})}{\lambda_k [2\lambda_k - (a_{11} + a_{22})]}, \quad (4b)$$ 

and 

$$S_r(\lambda_k, a_{12}) = S_r(\lambda_k, a_{21}) = \frac{a_{12}a_{21}}{\lambda_k [2\lambda_k - (a_{11} + a_{22})]}, \quad k = 1, 2. \quad (4cd)$$ 

(They evidently satisfy to the basic sensitivity invariant condition [2], i.e. \( \sum S_r(\lambda_k, a_{ij}) = 1 \)). 

3 Generalized conditions

Utilizing the basic formulas (3a,b), the sum of the relative sensitivity squares is obtained in the form 

$$\sum S_r^2(\lambda_k, a_{ij}) =$$
\[
\frac{\partial}{\partial a_{11}} \sum S_i^2(\lambda_k, a_{ij}) = 0 , \quad (6a)
\]
\[
\frac{\partial}{\partial a_{22}} \sum S_i^2(\lambda_k, a_{ij}) = 0 , \quad (6b)
\]
\[
\frac{\partial}{\partial a_{12}} \sum S_i^2(\lambda_k, a_{ij}) = 0 , \quad (6c)
\]
\[
\text{and} \quad \frac{\partial}{\partial a_{21}} \sum S_i^2(\lambda_k, a_{ij}) = 0 . \quad (6d)
\]

Starting from the (6a,b), the general optimized parameters \(a_{11}\) and \(a_{22}\) are obtained as
\[
a_{11} = a_{22} = \frac{1}{2}(\lambda_1 + \lambda_2) \quad (7)
\]
that corresponds to the basic design formula (3a), while the conditions (6c,d) finally entails the common generalized formula for the product of parameters \(a_{12}\) and \(a_{21}\)
\[
a_{12} a_{21} = \frac{1}{4}(\lambda_1 - \lambda_2)^2 \quad (8)
\]
that evidently corresponds to the basic design formula (3b).

**4 Detailed results for the individual special cases**

In the second-order systems three basic cases of two eigenvalues can exist:

(i) **Two different real eigenvalues** \((\lambda_1 \neq \lambda_2)\), where the corresponding state matrix can be rewritten into the form [3]
\[
A = \frac{1}{2} \begin{bmatrix}
\lambda_1 + \lambda_2 & (\lambda_1 - \lambda_2)K \\
(\lambda_1 - \lambda_2)K^{-1} & \lambda_1 + \lambda_2
\end{bmatrix}
\quad (9)
\]
and the coefficient \(K\) is a free parameter utilizable in optimized design procedure also for PWL dynamical systems of Class \(C\). The corresponding eigenvalue sensitivities and the resultant optimized sensitivity measures are summarized in Table 1.

(ii) **Two identical real eigenvalues** \((\lambda_1 = \lambda_2 = \lambda)\), where the corresponding state matrix becomes to the simplest Jordan form
\[
A = \begin{bmatrix}
\lambda & 0 \\
0 & \lambda
\end{bmatrix} . \quad (10)
\]
The simplified eigenvalue sensitivities and sensitivity measures are:
\[
S_r(\lambda, a_{11}) = S_r(\lambda, a_{22}) = S_r(\lambda, a_{11}) = S_r(\lambda, a_{22}) = \frac{1}{2} , \quad (11a)
\]
\[
S_r(\lambda, a_{12}) = S_r(\lambda, a_{21}) = S_r(\lambda, a_{12}) = S_r(\lambda, a_{21}) = 0 , \quad (11b)
\]
\[
\sum S_r^2(\lambda, a_{ij}) = \sum S_r^2(\lambda, a_{ij}) = \frac{1}{2} . \quad (11c)
\]

(iii) **Two complex conjugate eigenvalues** \((\lambda_1 = \lambda' + j\lambda'', \lambda_2 = \lambda' - j\lambda'')\). Substitute them into general conditions (7) and (8), we obtain the special simplified forms
\[
a_{11} = a_{22} = \lambda' , \quad a_{12} a_{21} = -\left(\lambda''\right)^2 \quad (12)
\]
and the corresponding state matrix can be rewritten into the so-called complex decomposed form, including the optimization coefficient \(K\) [3]
\[
A = \begin{bmatrix}
\lambda' & \lambda''K \\
-\lambda''K^{-1} & \lambda'
\end{bmatrix} . \quad (13)
\]
In this case all the sensitivity functions are obtained in the complex form and the same functions, expressed separately for the
eigenvalues real and imaginary parts, can easily be derived. Then the optimum sensitivity measures are

$$\sum S^2_r(\lambda',a_y) = \sum S^2_r(\lambda'',a_y) = \frac{1}{2}. \quad (14)$$

5 Conclusion

The resultant general equations (7) and (8) include a degree of freedom for the optimized model design for all types of eigenvalues. (In PWL systems of Class C it is represented by additional optimization coefficient [3].) In case of higher-order systems such a direct design procedure is not possible and the block-decomposed state matrix can be used [4].

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