Abstract. A new method for designing a minimal order perfect functional observers for singular continuous-time linear systems is proposed. Necessary and sufficient conditions for the existence of the minimal order perfect functional observer are established. A procedure for computation of matrices of the observers is derived and illustrated by a numerical example.

Key-Words: Design, existence, minimal order, perfect functional observer, singular, linear, system.

1 Introduction
The design of functional observers that reconstruct (estimate) directly a linear function $Kx$ ($K$ is known) of the state vector $x$ of linear systems is a very important problem. Because the functional observers do not need to reconstruct all state variables their orders can be significantly less than those of state observers. This problem was first proposed by Luenberger in [13]. The design of observers for linear systems has been considered in many papers and books [1-4,6,12-18]. Recently new concepts of perfect observers and functional observers for singular and standard linear systems have been proposed [5,9]. The concept of perfect observer has been extended for singular 2-D linear systems in [7] and the concept of perfect functional observer for singular continuous-time linear systems has been extended in [8]. The problem of the order reduction of functional observers of linear systems has been extensively investigated by Tsui in [16-18]. In [16] a design method of minimal order functional observers for standard linear continuous-time systems has been proposed with upper bound of the order

$$\min (n,v_1 + \cdots + v_m)$$

and

$$\min (n - p, (v_1 - 1) + \cdots + (v_m - 1))$$

for strictly proper and proper observers, respectively, where $n, m, p$ and $v_i (i = 1, \ldots, p)$ are the system (plant) order, number of inputs, number of outputs and observability indexes, respectively.

The main subject of this paper is to present a new design method of minimal order perfect functional observers for singular continuous-time linear systems. To the best author’s knowledge the design of perfect functional observers for singular linear systems has not been considered yet. Necessary and sufficient conditions for the existence of the perfect functional observers for singular linear systems will be established and a procedure for computation of matrices of the observers will be derived.

2 Problem Formulation
Let $R^{n\times m}$ be the set of real matrices and $R^n = R^{n\times 1}$. Consider the singular continuous-time linear system

$$\begin{align*}
E\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}$$

(1a)

(1b)

where $x \in R^n$, $u \in R^m$ and $y \in R^p$ are the semistate, input and output vectors, respectively and $E, A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $\det E = 0$. 
It is assumed that
\[ \det[E_s - A] = 0 \] for some \( s \in \mathbb{C} \) (2)
and \( \text{rank } C = p \).

We are looking for a minimal order perfect functional observer described by the equations
\[ E_i \dot{z} = Fz + Gu + Hy, \ z \in R^r, \] (3a)
\[ E_i \in R^{r 	imes r}, \ \det E_i = 0 \]
\[ w = Lz + My, w \in R^q, L \in R^{q 	imes r}, M \in R^{q 	imes p} \] (3b)
that reconstruct exactly the given linear function
\[ Kx, K \in R^{q 	imes n} \] is given (4)
i.e.
\[ w(t) = Kx(t) \text{ for all } t > 0 \] (5)
The problem can be stated as follows. Given \( E, A, B, C \) and \( K \), find \( E_1, F, G, H, L \) and \( M \) of (3) such that (5) holds.

Solvability conditions for the problem will be established and a procedure for computation of the matrices of (3) will be derived.

3 Problem Solution

Let us define
\[ e := z - TEx, \ e \in R^r \] (6)
Using (6), (1) and (3) we may write
\[ E_i \dot{e} = E_i \dot{z} - E_i TEx = F(e + TEx) + Gu + \]
\[ + HCx - E_i TAx - E_i TBu = \]
\[ = Fe + (FTE_i - E_i TA + HC)x + (G - E_i TB)u \] (7)
For
\[ E_i TA = FTE_i + HC \] and \( G = E_i TB \) (8)
from (7) we obtain
\[ E_i \dot{e} = Fe \] (9)
We choose \( E_i \) and \( F \) so that
\[ \det[E_i, s - F] = \alpha \] (10)
(nonzero scalar independent of \( s \))
For example, we may choose
\[ E_i = \begin{bmatrix} I_{r-1} & 0 \\ 0 & 0 \end{bmatrix} \in R^{r 	imes r}, F = \begin{bmatrix} 0 & I_{r-1} \\ -\alpha & 0 \end{bmatrix} \in R^{r 	imes r} \] (11)
where \( I_k \) is the \( k \times k \) identity matrix.

It is easy to show that if the matrices \( E_i \) and \( F \) have the canonical form (11) then the equation (9) has the solution \( e(t) = 0 \) for \( t > 0 \).

From (6) it follows that \( z = TEx \) if and only if \( e(t) = 0, t > 0 \) and then from (1b), (3b) and (5) we obtain
\[ K = LTE + MC \] (12)
The equation \( K = MC \) has a solution \( M \) if and only if
\[ \text{rank } C = \text{rank} \begin{bmatrix} C \\ K \end{bmatrix} \] (13)
In this case \( T = 0 \) and the linear function (4) can be reconstructed by \( My \). In what follows it is assumed that the condition (13) is not satisfied.

Lemma 1. If the system (1) is controllable and observable then there exists a pair of non-singular matrices \( P, Q \in R^{n \times n} \) such that
\[ PEQ = \begin{bmatrix} I_{n-p} & 0 \\ 0 & 0 \end{bmatrix}, \]
\[ PAQ = \begin{bmatrix} \overline{A}_1 & \overline{A}_2 \\ I_{n-p} & 0 \end{bmatrix}, \]
\[ \overline{A}_1 \in R^{p \times (n-p)}, \overline{A}_2 \in R^{p \times p}, PB = \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix}, \]
\[ B_1 \in R^{p \times m}, B_2 \in R^{(n-p) \times m}, CQ = \begin{bmatrix} 0 & I_p \end{bmatrix} \] (14)
The proof of Lemma is similar one to the given in [10].

Without loss of generality it is assumed that the matrices \( E, A, B \) and \( C \) of (1) have the canonical forms (14).

Let
\[ C = \begin{bmatrix} 0 & I_p \end{bmatrix} \] (15)
and
\[ K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}, K_1 \in R^{q \times (n-p)}, K_2 \in R^{q \times p} \] (16)
Then from (12) for \( K_2 = M \) we obtain
\[ K_1 = [K_1, 0] = K - MC = LTE \] (17)
and
\[ K_1 = LT_1 \] (18)
where
\[ TE = T \begin{bmatrix} I_{n-p} & \vdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} T_1 & 0 \end{bmatrix}, T_1 \in R^{rx(n-p)} \] (19)

From (18) it follows that
\[ \text{rank } K_1 \leq \min(q,r) \] and if \( r \leq q \) then
\[ r = \text{rank } K_1. \]

Taking into account (15) from (8) we have
\[ \begin{bmatrix} E_iT_1 - F \end{bmatrix} \begin{bmatrix} I_{n-p} \\ 0 \end{bmatrix} = 0 \] (20)

and
\[ \begin{bmatrix} E_iT_1 - F \end{bmatrix} \begin{bmatrix} 0 \\ I_p \end{bmatrix} = H \] (21)

**Lemma 2.** Let the matrices \( E,A,B,C \) and \( E_i,F \) have the canonical forms (14) and (11), respectively. Then the equations (20) and (21) can be written in the forms
\[ \begin{bmatrix} E_iT_1 - F \end{bmatrix} \begin{bmatrix} A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} A_2 \end{bmatrix} \] (22)

and
\[ H = E_iT_1 \begin{bmatrix} A_3 \\ A_4 \end{bmatrix} \] (23)

where
\[ T_1 \in R^{rx(n-p)}, T_2 \in R^{rxp}, A \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} I_{n-p} \\ 0 \end{bmatrix}, \]
\[ A_i \in R^{(n-p)x(n-p)}, A \begin{bmatrix} 0 \\ I_p \end{bmatrix}, A_3 \in R^{(n-p)p}, \]
\[ A_4 \in R^{p(n-p)}, A_3 \begin{bmatrix} A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} A_2 \end{bmatrix}, A_1 \in R^{nxp} \]

**Proof.** Taking into account (24) and
\[ TE \begin{bmatrix} I_{n-p} \\ 0 \end{bmatrix} = T_1 \] from (20) we obtain the equality
\[ E_i[T_1, T_2] \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0 \] which is equivalent to
\[ (22). \]

Note that
\[ E_i \begin{bmatrix} 0 \\ I_p \end{bmatrix} = 0 \]

and
\[ H = E_iT_2 \begin{bmatrix} A_3 \\ A_4 \end{bmatrix} \]

If \( E_i \) and \( F \) have the canonical forms (11) then the \( r \)-th row of the matrices \( E_iT_1A_1 \) and \( E_iT_2A_2 \) is zero and from (22) it follows that the first row of \( T_1 \) should be zero since by assumption \( \alpha \neq 0 \). Thus \( \text{rank } T_1 \leq r-1 \) and to satisfy the condition \( \text{rank } T_1 \geq \text{rank } K_1 \) we have to choose
\[ r \geq \text{rank } K_1 + 1 \] (25)

**Lemma 3.** Let the matrices \( E,A,B,C \) of (1) and \( E_i,F \) have the canonical forms (14) and (11), respectively. Then it is always possible to choose the coefficient \( \alpha \) of \( F \) such that the equation (22) has unique solution \( T_1 \) for any given matrices \( A_i \) and \( D = E_iT_2A_2 \).

**Proof.** Let
\[ A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in R^{mp\times nq} \]

be the Kronecker product of the matrices
\[ A = [a_{ij}], B \in R^{m\times n} \] and \( B \in R^{p\times q} \).

Then the equation (22) can be written in the form [11]
\[ St = d \] (26)

where
\[ S = F \otimes I_r - E_i \otimes A_i^T, \]
\[ t : [t_1, t_2, \ldots, t_{r-1} \otimes d = [d_1, d_2, \ldots, d_r] \] (27)

and \( t_i(d_i) \) is the \( i \)-th row of the matrix \( T_1(D) \).

The equation (26) has unique solution \( t \) if and only if the matrix \( S \) is non-singular or equivalently if all its eigenvalues are nonzero. It is easy to show [11] that the eigenvalues \( \lambda_S \) of \( S \) are given by \( \lambda_F - \lambda_E \lambda_A \) where \( \lambda_F \), \( \lambda_E \) and \( \lambda_A \) are the eigenvalues of \( F,E \) and \( A \), respectively. If \( F \) has the form (11) then \( \det[I, \lambda - F] = \lambda^r + \alpha \). Thus, it is always possible to choose \( \alpha \) so that all \( \lambda_S \) are nonzero. □

By Kronecker-Capelli theorem the equation (18) has a solution \( L \) if and only if
The matrix $T_2$ is chosen so that the equation (18) has a solution $L$ for given $K_1$ and $T_1$.

Therefore, the following theorem has been proved.

**Theorem.** Let the system (1) be controllable and observable and its matrices have the canonical forms (14). Then there exists a perfect functional observer (3) of the minimal order $11 + \text{rank} K_1$ for (1) if and only if the condition (28) is satisfied.

If the assumptions of Theorem are satisfied then a perfect functional observer (3) of the order $r$ for (1) can be computed by the use of the following procedure.

**Procedure**

**Step 1.** Using the slightly modified method given in [11] transform the matrices $E,A,B,C$ to the canonical forms (14).

**Step 2.** Using (16) and (17) compute

$$M = K_2, K_1 = [K_1, 0] = K - MC$$

and

$$K = \begin{bmatrix} 0 & 1 & -1 & 1 & 3 \\ 0 & -2 & 2 & 2 & 4 \end{bmatrix}$$

It is easy to check that the system is controllable and observable.

In this case we have $n = 5, m = 1, p = 2$ and using Procedure we obtain

**Step 1.** The matrices (30) have already the canonical forms (14).

**Step 2.** Using (16) and (17) we obtain

$$M = K_2 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix},$$

$$K_1 = K - MC = [K_1, 0] = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 \end{bmatrix}$$

and

$$K_1 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix},$$

$$\text{rank} K_1 = \text{rank} \begin{bmatrix} 0 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix} = 1$$

we have $r = 2$.

**Step 3.** It is easy to check that for $\alpha = 2$ the matrix $S$ (defined by (27)) is non-singular. For $T = [t_g] \in R^{2 \times 5}$ the equation (22) has the form

$$G = E_i TB$$

compute the matrices $H$ and $G$.

**Step 6.** Using (3) find the desired perfect functional observer

**Remark.** If $\text{rank} K_1 = 1$ then by (23) and (29) $H = 0$ and $G = 0$ since $E_i T = 0$. Hence the equation (3a) takes the form $\dot{z} = Fz$.

**4 Example**

Design a perfect functional observer (3) for the system (1) with

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$K = \begin{bmatrix} 0 & 1 & -1 & 1 & 3 \\ 0 & -2 & 2 & 2 & 4 \end{bmatrix}$$

(31)
and its solution is
\[
T_i = \begin{bmatrix}
0 & 0 & 0 \\
0 & t_{14} & t_{15}
\end{bmatrix}
\]  
(32)

**Step 4.** The entries \(t_{14}, t_{15}\) of \(T_2\) are chosen so that the condition
\[
\text{rank} = \begin{bmatrix}
0 & 0 & 0 \\
0 & t_{14} & t_{15}
\end{bmatrix} = \text{rank} \begin{bmatrix}
0 & 0 & 0 \\
0 & t_{14} & t_{15} \\
0 & 1 & -1 \\
0 & -2 & 2
\end{bmatrix}
\]  
(33)
is satisfied. The condition (33) holds, for example, for \(t_{14} = -t_{15} = 1\).

In this case the equation (18) has the form
\[
\begin{bmatrix}
0 & 1 & -1 \\
0 & -2 & 2
\end{bmatrix} = L \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & -1
\end{bmatrix}
\]
and its solution is
\[
L = \begin{bmatrix}
l_1 & 1 \\
l_2 & -2
\end{bmatrix} \quad (l_1, l_2 \text{ are arbitrary})
\]

**Step 5.** Using (23), (29) and (32) we obtain \(H = 0\) and \(G = 0\).

**Step 6.** The desired perfect functional observer has the form
\[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} z = \begin{bmatrix}
0 & 1 \\
-2 & 0
\end{bmatrix} z
\]
\[
w = \begin{bmatrix}
l_1 & 1 \\
l_2 & -2
\end{bmatrix} \begin{bmatrix}
z \end{bmatrix} + \begin{bmatrix}
1 & 3 \\
2 & 4
\end{bmatrix} y
\]

5 Concluding remarks

A new design method of minimal order perfect functional observers for singular continuous-time linear systems has been proposed. Necessary and sufficient conditions for the existence of the minimal order perfect functional observer have been established. A procedure for computation of matrices of the perfect functional observer (3) has been derived and illustrated by a numerical example. If the condition (13) is satisfied then the linear function (4) can be reconstructed by \(My\) and the matrix \(M\) can be found from the equation \(K = MC\).

With minor modifications the considerations can be also applied to singular discrete-time linear systems. An open problem is an extension of the considerations for two-dimensional linear systems [6,7].

References


