A SINGULAR VALUE DECOMPOSITION BASED KALMAN FILTERING FOR CONTROL OF IRRIGATION CANALS

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Abstract: - A discrete-time linear Kalman filtering algorithm based on singular value decomposition (SVD) technique is applied to the control of irrigation canals. The Saint-Venant equations of open-channel flow are linearized using the Taylor series and a finite difference approximation of the original nonlinear, partial differential equations. Using the linear optimal control theory, a Linear Quadratic Gaussian (LQG) controller is designed for an irrigation canal with 6 gates and 5 pools. The Kalman state estimator in the LQG controller is designed using both conventional and singular value decomposition (SVD) techniques. The results from both techniques are compared to observe the performance of state variables estimation for an irrigation canal. The singular value decomposition (SVD) based Kalman filter formulation has a good numerical stability and can handle correlated measurement noise without any additional transformation. In conventional Kalman filter applications, measurements are generally designed to be uncorrelated; and this may be a large source of error. The results of this study show that a SVD based Kalman filter for irrigation canals offers an alternative to conventional Kalman filter when the flow depth measurements in the canal are correlated.

Keywords: - Singular value decomposition (SVD), kalman filters, open-channel flow.

1 Introduction
Conveyance and distribution performance of irrigation canals can be improved to better meet the requirements of farmers by providing modern methods of canal control. The demand for irrigation water varies with time among other factors, due to the variable weather conditions. Therefore, to avoid overflows and always be able to satisfy the demand the canal system must be controlled to maintain desired flow rates and water surface elevations. The use of flexible irrigation deliveries is necessary for efficient on-farm irrigation water management. Thus, the conversion from rigid to flexible delivery schedules will require better canal control to provide good uniform deliveries. The strategy of water delivery, the desired schedules, required communications (both digital and human), and the location of a variety of decision making must first be decided upon before the hardware is examined. The control strategy must be compatible with the flexibility of the ultimate water supply, and the social, political, geographical, and economic conditions under which it will be used [1]. Furthermore, the control strategy used in automating canals for irrigation water
delivery.
delivery has important effects upon social development and the economic well-being of the irrigation project. When the needs of farmers are more closely matched by canal deliveries, on-farm water management potentials are greater. Canal flows must match turnout demands and water levels must remain high enough to supply all turnouts plus steady enough to maintain a constant delivery through gravity turnouts into open laterals. Additionally, canals must be operated to prevent rapid water level drawdown that can damage canal lining.

The principal objectives of irrigation canal networks are to provide equitable and timely apportionment of water among irrigators and to do so in proper quantities (flow rate and duration). To accomplish these objectives, canal should be designed and operated to minimize flow rate and depth fluctuations in order to protect system structures, minimize required management activities, and reduce seepage and losses [9]. The prerequisites for such operational efficiency include appropriate and properly maintained water conveyance and control structures, adequate water measurement and communication systems, and proper coordination of all operation and maintenance activities. The goal of the canal operations is to match the actual flow in the canal to the required flow for that day while maintaining water surface elevations within allowable limits. In a demand delivery schedule, whether under constant level control or constant volume control, since the demands are not known in advance, the effect of the random disturbances on the system variables must be measured and used in the feedback loop to control the system. In the past, significant research was done to derive the relationship between the deviations in the system state variables (flow rate and flow depth) and the change in gate opening. Recently, Reddy [10], Malaterre [6], Reddy [11] have applied optimal control theory to derive control algorithms for modernization of irrigation canals. However, when lumped parameter models are used to derive control algorithms for irrigation canals, the number of state variables that must be used in the feedback loop is large [9]. It is very expensive to measure all state variables (flow depth and flow rates) in a multi-pool irrigation canal. [11] applied a Kalman filtering technique in the control of irrigation canals. However, he assumed the random measurement (flow depth) noise is a white, zero-mean Gaussian random sequence, in other words the measurements are uncorrelated. The objective of this paper is to apply a singular value based (SVD) based Kalman filter technique in the control of an irrigation canal, where flow depth measurement noises are correlated, and to observe the performance of LQG regulators which are designed using both conventional Kalman filtering technique and SVD technique.

2 Modeling of Open-channel Flow

In the operation of irrigation canals, decisions regarding the changes in gate opening in response to arbitrary (random) changes in the water withdrawal rates into lateral or branch canals are required to maintain the flow rate into the laterals close to the desired value. This is accomplished by maintaining the depth of flow or the volume of water in a given pool at a target value. This problem is similar to the process control problem in which the state of the system is maintained close to
the desired value by using real-time feedback control. The Saint-Venant equations, presented below, are used to model flow in a canal:

\[
\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q_i
\]  

(1)

\[
\frac{\partial Q}{\partial t} + \frac{\partial (Q^2 / A)}{\partial x} + gA\frac{\partial y}{\partial x} - S_o + S_f = 0
\]  

(2)

in which \( A \) = wetted area, \( m^2 \); \( q_i \) = lateral flow, \( m^2/sec \); \( y \) = water depth, \( m \); \( t \) = time, \( sec \); \( x \) = longitudinal direction of channel, \( m \); \( g \) = gravitational acceleration, \( m^2/sec \); \( S_0 \) = canal bottom slope (\( m/m \)); \( R \) = hydraulic radius, \( A/P \) (\( m \)); \( P \) = wetted perimeter (\( m \)); \( n \) = roughness coefficient (\( s/m^{1/3} \)); and \( S_f \) = the friction slope, \( m/m \), and is defined as

\[ S_f = Q|Q|/K^2 \]

in which \( K \) = hydraulic conveyance of canal = \( AR^{2/3}/n \); \( R \) = hydraulic radius, \( m \); and \( n \) = Manning friction coefficient, \( s/m^{1/3} \). In deriving Eq. (2), the effect of the net acceleration terms stemming from removal of a fraction of the surface stream was assumed negligible [5].

Lateral canals in the main canal are usually scattered throughout the length of the supply canal. Manually controlled discharge regulators are used at the head of a lateral canals: The mathematical representation of flow through these structures is given as follows:

\[ q_i = C_d b_l w_l (2g(Z-Z_l))^{1/2} \]  

for submerged flow  

(3)

\[ q_i = C_d b_l w_l (2g(Z-E_s))^{1/2} \]  

for free flow  

(4)

in which, \( q_i \) = lateral discharge rate, \( m^3/s \); \( C_d \) = outlet discharge coefficient; \( b_l \) = width of outlet structure, \( m \); \( w_l \) = height of gate opening of outlet structure, \( m \); \( Z \) = water surface elevation in supply canal, \( m \); and \( Z_l \) = water surface elevation in lateral canal, \( m \); and \( E_s \) = sill elevation of head regulator, \( m \). Obviously, the flow rate through a head regulator depends upon the water surface elevation in the supply canal. The water surface elevation in the lateral canal is a function of the discharge rate through the head regulator. Therefore, this equation is an implicit equation. In the case of free flow, the discharge rate through the head regulator is independent of the water surface elevation in the lateral canal. Therefore, once the required discharge into a lateral is specified, then the gate opening is adjusted to get the required flow rate through the head regulator, assuming that the water surface elevation in the supply canal is maintained constant at the target level. When a manually controlled head regulator is used, for simulation purposes the gate opening or the variation in gate opening is specified as a function of time [4]. Conversely when an automated discharge rate regulator is used, for simulation purposes the lateral discharge rate as a function of time is specified as a known input, i.e., \( q_i = f_d(t) \).

In the regulation of the main canal, decisions regarding the opening of gates in response to random changes in water withdrawal rates into lateral canals are required to maintain the flow rate into laterals close to the desired value. This is accomplished by either maintaining the depth of flow in the immediate vicinity of the turnout structures in the supply canal constant or by maintaining the volume of water in the canal pools at the target value. When the latter option is used, the outlets are often fitted with discharge rate regulators. The water levels or the volumes of water stored in the canal...
pools are regulated using a series of spatially distributed gates (control elements). Hence, irrigation canals are modeled as distributed control systems. Therefore, in the solution of Eqs. (1) and (2), additional boundary conditions are specified at the control structures in terms of the flow continuity and the gate discharge equations, which are given by:

\[ Q_{i-1,N} = Q_{i,l} \quad \text{(continuity)} \quad (5) \]
\[ Q_{gi} = C_{di} b_i u_i (2g(Z_{i-1,N} - Z_{i,l}))^{1/2} \quad \text{(gate discharge)} \quad (6) \]

in which, \( Q_{i-1,N} \) = flow rate through downstream gate (or node N) of pool \( i-1 \), m³/sec; \( Q_{gi} \) = flow rate through upstream gate (or node 1) of pool \( i \), m³/sec; \( C_{di} \) = discharge coefficient of gate \( i \); \( b_i \) = width of gate \( i \), m; \( u_i \) = opening of gate \( i \), m; \( Z_{i-1,N} \) = water surface elevation at node \( N \) of pool \( i-1 \), m; \( Z_{i,l} \) = water surface elevation at node 1 of pool \( i \), m; and \( i \) = pool index (\( l = 0 \) refers to the upstream constant level reservoir).

2.1 Linearization and discretization of system equations

Linear control theory is well developed and is easier to apply than nonlinear control theory. The Saint-Venant open-channel equations are linearized about and average operating condition of the canal to apply the linear control theory concepts to the problem. After applying a finite-difference approximation and the Taylor series expansions to Eqs. (1) and (2), a set of linear, ordinary differential equations is obtained for the canal with control gates and turnouts:

\[ A_{11} \delta Q_{j}^+ + A_{12} \delta z_j + A_{13} \delta Q_{j+1}^+ + A_{14} \delta z_{j+1} = A'_{11} \delta Q_{j} + A'_{12} \delta z_j + A'_{13} \delta Q_{j+1} + A'_{14} \delta z_{j+1} + C_1 \]
\[ = A_{21} \delta Q_{j}^+ + A_{22} \delta z_j + A_{23} \delta Q_{j+1}^+ + A_{24} \delta z_{j+1} = A'_{21} \delta Q_{j} + A'_{22} \delta z_j + A'_{23} \delta Q_{j+1} + A'_{24} \delta z_{j+1} + C_2 \]

where \( \delta Q_{j}^+ \) and \( \delta z_j \) = discharge and water-level increments from time level \( n+1 \) at node \( j \); \( \delta Q_{j} \) and \( \delta z_j \) = discharge and water-level increments from time level \( n \) at node \( j \); and \( A_{11}, A'_{21}, \ldots, A_{12}, A_{22} \) are the coefficients of the continuity and momentum equations, respectively, computed with known values at time level \( n \). Similar equations are derived for channel segments that contain either a lateral, a gate structure, a weir or other type of hydraulic structure. The matrix form of the above equations for the canal can be defined as follows:

\[
\begin{bmatrix}
A_1 & A_2 & A_3 & A_4 \\
A_{11} & A_{12} & A_{13} & A_{14} \\
- (\frac{\partial f}{\partial z_j}) & 1 - (\frac{\partial f}{\partial z_{j+1}})
\end{bmatrix}
\begin{bmatrix}
\delta Q_j^+ \\
\delta z_j \\
\delta Q_{j+1}^+ \\
\delta z_{j+1}
\end{bmatrix} =
\begin{bmatrix}
A'_{11} & A'_{12} & A'_{13} & A'_{14} \\
A'_{11} & A'_{12} & A'_{13} & A'_{14} \\
- (\frac{\partial f}{\partial z_j}) & 1 - (\frac{\partial f}{\partial z_{j+1}})
\end{bmatrix}
\begin{bmatrix}
\delta Q_j \\
\delta z_j \\
\delta Q_{j+1} \\
\delta z_{j+1}
\end{bmatrix} +
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
\frac{\partial f}{\partial z_j}
\end{bmatrix}
\delta \Delta u +
\begin{bmatrix}
A_{21} & A_{22} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
\delta Q_p \\
\delta Q'_p
\end{bmatrix}
\]
From matrix form of the equations above, the state of system equation at any sampling interval \( k \) can be written, in a compact form as follows:

\[
A_L \delta x(k+1) = A_R \delta x(k) + B \delta u(k) + C \Delta \delta q(k)
\] (10)

where \( A = l \times l \) system feedback matrix, \( B = l \times m \) control distribution matrix, \( k \) = time increment, sec; \( \Delta q \) = variation in demands (or disturbances) at the turnouts, \( m^2/s \). The elements of the matrices \( A, B, \) and \( C \) depend upon the initial condition. The Eq. (10) can be written in a state-variable form along with the output equations as follows:

\[
\delta x(k+1) = \Phi \delta x(k) + \Gamma \delta u(k) + \Psi \delta q(k)
\] (11)

\[
\delta y(k) = H \delta x(k) + \eta(k)
\] (12)

where \( \Phi = (A_L)^{-1} \star A_R \), \( \Gamma = (A_L)^{-1} \star B \), and \( \Psi = (A_L)^{-1} \star C \), \( \delta x(k) = l \times 1 \) state vector, \( \delta u(k) = m \times 1 \) control vector, \( \delta q(k) = p \times 1 \) matrix representing external disturbances (changes in water withdrawal rates) acting on the system, \( \delta y(k) = r \times 1 \) vector of output (measured variables), \( H = r \times l \) output matrix, \( l \) = number of dependent (state) variables in the system, \( m \) = number of controls (gates) in the canal, \( p \) = number of outlets in the canal, \( \eta(k) \) = measurement noise, and \( r \) = number of outputs. The elements of the matrices \( \Phi, \Gamma, \) and \( \Psi \) depend upon the canal parameters, the sampling interval, and the assumed average operating condition of the canal.

In equation (10), the vector of state variables is defined as follows:

\[
\delta x = (\delta Q_{i,1}, \delta Z_{i,2}, \delta Q_{i,2}, \ldots, \delta Z_{i,N-1}, \delta Q_{i,N-1}, \delta Q_{i,N})
\] (13)

3 Design of Linear Quadratic Gaussian Controller (LQG)

The LQG theory provides an integrated knowledge base for the development of a flexible controller. The LQG controller integrates the states estimation and the controller design into a single body of knowledge. A LQG controller consists of an optimal state feedback (LQR) and an optimal state estimator (Kalman filter). An optimal LQG controller based upon a linear system, a quadratic objective function and an assumption of white noise that has a normal, or Gaussian, probability distribution. In short, the optimal LQG design process is the following: a) Design an optimal regulator for a linear system assuming full-state feedback (i.e. assuming all the state variables are available for measurement) and a quadratic objective function. The regulator is designed to generate a control input, \( \delta u(k) \), based upon the measured state vector \( \delta x(k) \). b) Design a Kalman filter for the system assuming a known control input, \( \delta u(k) \), a measured output, \( \delta y(k) \), and white noises, \( \eta(k) \) and \( \delta q(k) \), with known power spectral densities, \( \rho \) [8]. The Kalman filter is designed to provide an optimal estimate of the state vector, \( \delta x(k) \). c) Combine the separately designed optimal regulator and Kalman filter into an optimal compensator, which generates the input vector, \( \delta u(k) \), based upon the estimated state-vector, \( \hat{x}(k) \), rather than the actual state-vector, \( \delta x(k) \), and the measured output vector, \( \delta y(k) \).

3.1 Design of Linear Quadratic Regulator (LQR)

LQR control problem as an optimization problem in which the cost function, \( J \), to be minimized is given as follows:
\[ J = \sum_{i=1}^{K_{\infty}} (\delta x^T(k)Q_{x_{\infty}} \delta x(k) + \delta u^T(k)R_{\infty} \delta u(k)) \]  

subject to the constraint that:

\[ -\delta x(k+1) + \Phi \delta x(k) + \Gamma \delta u(k) = 0 \quad k = 0, \ldots, K_{\infty} \]  

where \( K_{\infty} \) = number of sampling intervals considered to derive the steady state controller; \( Q_{x_{\infty}} \) = state cost weighting matrix; and \( R_{\infty} \) = control cost weighting matrix. The matrices \( Q_x \) and \( R \) are symmetric, and to satisfy the non-negative definite condition, they are usually selected to be diagonal with all diagonal elements positive or zero. The first term in Eq. (14) represents the penalty on the deviation of the state variables from the average operating (or target) condition, where the second term represents the cost of control. This term is included in an attempt to limit the magnitude of the control signal \( \delta u(k) \). Unless a cost is imposed for use of control, the design that emerges is liable to generate control signals that cannot be achieved by the actuator. In this case the saturation of the control signal will occur resulting in a system behavior that is different from the closed loop system behavior that was predicted assuming that saturation will not occur. Therefore, the control signal weighting matrix elements are selected to be large enough to avoid saturation of the control signal under normal operating conditions. Eqs. (14) and (15) constitute a constrained-minimization problem that can be solved using the method of Lagrange multipliers. This produces a set of coupled difference equations which must be solved recursively backwards in time. In the optimal steady-state case, the solution for change in gate opening, \( \delta u(k) \), is of the same form as:

\[ \delta u(k) = -K \delta x(k) \]  

where \( K \) is given by:

\[ K = [R + \Gamma^T \Sigma \Gamma]^{-1} \Gamma^T S \Phi \]  

S is a solution of the discrete algebraic Riccati equation (DARE)

\[ \Phi^T S \Phi - \Phi^T S \Sigma [R + \Gamma^T \Sigma \Gamma]^{-1} \Gamma^T S \Phi + Q_x = S \]  

where \( R = R^T > 0 \) and \( Q_x = Q_x^T = H^T H \geq 0 \). The control law defined by Eq. (16) brings an initially disturbed system to an equilibrium condition in the absence of any external disturbances acting on the system. In hydraulic engineering problems, the depth of flow, flow rate, and velocity as a function of distance can be considered as the state or internal variables. Sometimes, the volume of water in a given reach of a canal can also be considered as a state variable. In this paper, the water surface elevation and flow rate were considered the state variables. Given initial conditions \( [\delta x(0)] \), \( \delta u \), and \( \delta q \), Eq. 16 can be solved for variations in flow depth and flow rate as a function of time. If the system is really at equilibrium [i.e., \( \delta x(0) = 0 \) at time \( t = 0 \)] and there is no change in the lateral withdrawal rates (disturbances), the system would continue to be at equilibrium forever; then, there is no need for any control action. Conversely, in the presence of disturbances (known or random), the system would deviate from the equilibrium condition. The actual condition of the system may be either above or below the equilibrium condition, depending upon the sign and magnitude of the disturbances. If the
system deviates significantly from the equilibrium condition, the discharge rates into the laterals will be different (either more or less) than the desired values. But in canal operations, the main objective is to keep these deviations to a minimum so that a nearly constant rate of discharge is maintained through the turnouts [3].

3.2 Design of Conventional Kalman filter
In the past three decades linear optimal filtering has been mostly based on minimization of the variance of the estimation error. The well known example of this approach is the Kalman filter. By using discrete-time model, conventional Kalman filter facilitates the estimation of the unknown state vector recursively for each new observation [5]. It is considered an optimal estimator because it provides a minimum variance estimate. Since it is expensive to measure all the state variables (flow rates and flow depths) in a canal system, the number of measurements per pool must be kept to an absolute minimum. Usually the flow depths at the upstream and downstream ends of each pool are measured. For steady-state Kalman filter, the observer gain matrix, $L$, is calculated as follows:

$$L = PH^T [HPH^T + R_i]^{-1}$$

(19)

where $P$ is the covariance of estimation uncertainty:

$$
\begin{align*}
\Phi^T P \Phi &+ PH^T [R_i + H^T P H]^{-1} H P \\
\Phi^T + Q_i &= P
\end{align*}

(20)

where $R_i = R_i^T > 0$ is a tolerance values for the $R_i$ covariance matrix which is an identity matrix and $Q_i = Q_i^T \geq 0$ is a diagonal matrix. The disturbances $\delta q(k)$ and $\eta(k)$ (in Eqs. (11) and (12)) are assumed to be zero mean Gaussian white noise sequences with symmetric positive definite covariance matrices $Q_i$ and $R_i$, respectively. Furthermore, sequences $\delta q(k)$ and $\eta(k)$ are assumed to be statistically independent [14]. The system dynamic equation is used to predict the state and estimation error covariance as follows: time update equations:

$$P(k+1)^\dagger = \Phi P^\dagger \Phi^T + \Psi Q_i \Psi^T$$

(21)

$$\dot{\xi}(k+1)^\dagger = \Phi \dot{\xi}(k) + \Gamma^\dagger \delta u(k)$$

(22)

in which $\dot{\xi}(k)$ = estimated values of the state variables and superscript $\dagger$ refers to values after the measurement update. As soon as measured values for the output variables $\delta y(k)$ are available, the time-update values are corrected using the measurement update equations as follows: measurement update equations:

$$L(k+1) = P(k+1)^\dagger H^T [HP(k+1)^\dagger H + R_i]^{-1}$$

(23)

$$P(k+1)^\dagger = P(k+1)^\dagger - L(k+1) HP(k+1)^\dagger$$

(24)

$$\dot{\xi} (k+1)^\dagger = \dot{\xi} (k+1)^\dagger + L(k+1)[\delta y(k+1) - H \dot{\xi} (k+1)^\dagger]$$

(25)

If the initial conditions and the inputs (control inputs and the disturbances) are known without error, the system dynamic Eq. (2) can be used to estimate the state variables that are not measured. Since part of the disturbances are random and usually are not measured, the canal parameters are not known very accurately, the estimated values of the state variables would diverge from the actual values. This divergence can be minimized by utilizing the difference between measured output and the estimated output (error signal), and by constantly correcting the system model.
with the error signal [4]. Therefore, the modified state equations are given as:

\[ \dot{\delta}(k+1) = \Phi \delta(k) + \Gamma \delta u(k) + L \left[ \delta y(k) - H \delta \hat{x}(k+1) \right] \]  

(26)

3.3 Design of Singular Value Decomposition Based Kalman filter

The major disadvantage of the conventional Kalman formulation is that the matrix subtraction in Eq. 24, representing the reduction in uncertainty due to the measurement, can yield a result \( P(k+1) \) that is computationally not positive definite. To circumvent this difficulty, the idea of using a square root of the covariance matrix in the algorithmic implementation was introduced [14]. This is a matrix:

\[ P = SS^T \]  

(27)

where \( S \) is obtained in triangular form by the well-known Cholesky decomposition. Although equivalent algebraically to the conventional Kalman filter recursion, the square root approach exhibits improved numerical precision and stability [14]. The singular-value decomposition (SVD) is one of the most useful tools in linear algebra and has been widely used in control theory during the last three decades. The SVD of an \( m \)-by-\( n \) matrix \( A \) (\( m \geq n \)), is a factorization of \( A \) into a product of three matrices. That is, there exist orthogonal matrices \( U \) and \( V \) such that

\[ A = U \Lambda V^T \]  

(28)

Where \( \Lambda \) is a quasidiagonal matrix with singular values as its diagonal elements; that is

\[ \Lambda = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \]  

(29)

where \( S \) is a diagonal matrix of nonzero singular values in descending order:

\[ S = \text{diag}(\sigma_1, \ldots, \sigma_r), \sigma_1 \geq \ldots \geq \sigma_r > 0 \]  

(30)

The numbers \( \sigma_1 \ldots \sigma_r \) together with \( \sigma_{r+1} = 0, \ldots, \sigma_n = 0 \) are called the singular values of \( A \) and they are the positive square roots of the eigenvalues of \( A^T A \). The columns of \( U \) are called the left singular vectors of \( A \) (the orthonormal eigenvectors of \( A A^T \)) while the columns of \( V \) are called the right singular vectors of \( A \) (orthonormal eigenvectors of \( A^T A \)). It is known that the singular values and singular vectors of a matrix are relatively insensitive to perturbations in the entries of the matrix, and to finite precision errors [14]. In practice, if \( A^T A \) is positive definite then Eq. 28 can be reduced to

\[ A = U \begin{bmatrix} S \\ 0 \end{bmatrix} V^T \]  

(31)

Especially, if \( A \) itself is symmetric positive definite then we will have a symmetric singular value decomposition

\[ A = USU^T = UD^2 U^T \]  

(32)

In the covariance Eq. 21 of the conventional Kalman filter, assume that the singular value decomposition of \( P \) is available and has been propagated and update by the filter algorithm. Thus we have

\[ P^+ = U^+ D^+ 2 U^{+T} \]  

(33)

Therefore, Eq. 21 can be written as

\[ P(k+1) = \Phi U^+ D^+ 2 U^{+T} \Phi^T + \Psi Q_1 \Psi^T \]  

(34)

The purpose is to find the factors \( U(k+1) \) and \( D(k+1) \) from Eq. 34 such that

\[ P(k+1) = U(k+1) D^2(k+1) U^T(k+1), \]
where $U$ factors are orthogonal and $D$ factors are diagonal. Provided that there is no danger of numerical accuracy deterioration, one could compute $P(k+1)$ and then apply the singular value decomposition of symmetric positive definite matrix which is given by Eq. 32. However, it has been shown that this is not a good numerical exercise [14]. Instead it can be defined as $(s+n)$-by-$n$ matrix

$$
\begin{pmatrix}
D & \Phi^T \\
(Q^{1/2})^T & \Psi^T
\end{pmatrix}
$$

and compute its singular value decomposition

$$
\begin{pmatrix}
D & \Phi^T \\
(Q^{1/2})^T & \Psi^T
\end{pmatrix} = U \begin{pmatrix} D' \\ 0 \end{pmatrix} V^T
$$

Multiplying each side on the left by its transpose, we have

$$
\Phi U^T D^T D^+ U^T \Phi + \Psi (Q_2^{1/2})^T \Psi^T = V^T \begin{pmatrix} D^T \\ 0 \end{pmatrix} U^T U \begin{pmatrix} D' \\ 0 \end{pmatrix} V^T
$$

and finally time update equation can be written as

$$
\Phi U^T D^{1/2} U^T \Phi + \Psi (Q_2^{1/2})^T \Psi^T = V' D^2 V^T
$$

Comparing the result with Eq. 34, $V'$ and $D'$ are just the $U(k+1)$ and $D(k+1)$. In the conventional Kalman measurement update, substituting Eqs. (23) into (24) yields

$$
P(k+1)^+ = P(k+1)^- - P(k+1)^- H H^T [HP(k+1) H^T + R]^{-1} H P(k+1)^-
$$

Applying the singular value decomposition of symmetric positive definite matrix to $P(k+1)$ and $P$, respectively,

$$(U^T D^{+2} U^T)^{-1} = (UD^T U^T)^{-1} + H^T R^{-1} H
(\bar{U}^T)^{-1} (D^2 + U^T H^T H U) \bar{U}^T$$

In Eq. 40 let

$$R^T = LL^T$$

be the Cholesky decomposition of the inverse of the covariance matrix. If the inverse is available then there is no difficulty. If the covariance matrix $R$ is known, but not its inverse, then the reverse Cholesky decomposition $R^{1/2} R^{1/2} = R_\nu$, $R^{1/2}$ upper triangular, can be found. It then follows that $(L = R^{1/2} T)^{-1}$ is the required Cholesky decomposition in Eq. 41 [14]. Now considering the $(m+n)$-by-$n$ matrix

$$
\begin{pmatrix}
L^T H U \\
D^{-1}
\end{pmatrix}
$$

and computing its singular value decomposition

$$
\begin{pmatrix}
L^T H U \\
D^{-1}
\end{pmatrix} = U^T \begin{pmatrix} D^T \\ 0 \end{pmatrix} V^T
$$

Multiplying each side on the left by its transpose yields

$$D^2 + U^T H^T L L^T H U = V^T D^{+2} V^T
$$

Then the Eq. (40) can be written as follows

$$\begin{pmatrix} (U^T)^{-1} (D^+)^{-2} (U^T)^{-1} - (U^T)^{-1} V^+ D^{+2} V^+ U \\
1 - [(UV^+)^T]^{-1} D^2 [(UV^+)^T]
\end{pmatrix}
$$

Comparing tow sides of Eq. (45)

$$U^+ = U V^+
$$

$$D^+ = (D^+)^{-1}$$
In this manner, a new measurement update formulation has been obtained. The crucial component of the update involves the computation of the singular values decomposition of an unsymmetric matrix – without explicitly forming its left orthogonal factor that has a high dimension [14]. For the Kalman gain an alternative expression may also be derived as follows:

\[ L = P^+ H^T R^{-1} = U^+ D^+ U^T H^T LL^T \] (48)

The state variables measurement update is given by

\[ \delta^+ (k+1) = \delta^+ (k+1)^- + L(k+1)[ \delta y(k+1) - H \delta^+ (k+1)^- ] \] (49)

4 Results and Discussions

An example problem is considered to compare and demonstrate the performance of Kalman filtering design techniques discussed earlier. Both conventional and singular value decomposition Kalman filter based LQG controller was applied to a 38 km-long irrigation canal. The canal includes 6 gates to maintain constant water levels immediately upstream of themselves. The irrigation canal has 5 pools and each pool has its canal geometry, gate and turnout properties. For numeric solution purposes, each pool was divided into nodes and all 5 pools together have 141 nodes. The downstream flow requirement at the end of the canal is 44.4 m$^3$/sec, and the opening of the last gate in the system was fixed at its initial steady state value. The target water levels, the initial flow rates into the lateral canals, and the disturbances are given in Table 1. Using these initial values for 5 pools, a state-variable model with 271 state variables was formulated.

The state variable equation was supplemented with an output equation, in which the output variables are the flow depths and the upstream and downstream ends of the pools. The variations in the depth of flow at the downstream end of the pools were the controlled variable, and the control objective is regulate the flow depth at the downstream end of the pools constant in the presence of disturbances acting on the system.

Table 1. Initial Data Used in Simulation Study

<table>
<thead>
<tr>
<th>Pool Number</th>
<th>Turnout initial flow rate (m$^3$/s)</th>
<th>Target Depth (m)</th>
<th>Disturbances (m$^3$/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.398</td>
<td>5</td>
<td>2.5</td>
</tr>
<tr>
<td>2</td>
<td>11.807</td>
<td>4.7</td>
<td>2.5</td>
</tr>
<tr>
<td>3</td>
<td>0.63</td>
<td>4.6</td>
<td>2.5</td>
</tr>
<tr>
<td>4</td>
<td>4.314</td>
<td>4.4</td>
<td>2.5</td>
</tr>
<tr>
<td>5</td>
<td>11.097</td>
<td>4.1</td>
<td>2.5</td>
</tr>
</tbody>
</table>

First LQG controller with conventional Kalman filter is simulated. Measurement noise has been assumed to be uncorrelated. At the end of the simulation (15000 sec), the flow depth variations are -0.0166 m, -0.0369 m, -0.0519 m, -0.0442 m, and -0.1097 m in pool1, pool2, pool3, pool4 and pool5, respectively (Figure 1). The changes in gate opening, at the end of the simulation, are coming close to equilibrium level. The incremental gate openings are 0.0003 m, 0.0003 m, 0.0002 m, 0.0002 m, and 0.0003 m at gate1, gate2, gate3, gate4 and gate5, respectively (Figure 2). After the simulation, the LQG controller is redesigned with singular value decomposition based Kalman filter and measurement noise has been assumed to
be correlated. The algorithm is again simulated and the flow depth variations are -0.0170 m, -0.0382 m, -0.0544 m, -0.0470 m, and -0.1189 m in pool1, pool2, pool3, pool4, and pool5, respectively (Figure 1). If these flow depth variations are compared to conventional Kalman filter, there are no significant differences between the numbers. At the end of 15000 seconds, the changes in gate openings are approaching to equilibrium position. The incremental gate openings are 0.0003 m, 0.0003 m, 0.0002 m, 0.0002 m, and 0.0003 m at gate1, gate2, gate3, gate4, and gate5 respectively (Figure 2). Changes in gate openings don’t have significant difference in both techniques. Conventional Kalman filter design formulation doesn’t have the ability to handle correlated measurements noise without an additional transformations and this may be a large source of error. Singular values decomposition based Kalman filter formulation can efficiently overcome this defect without loosing its performance.

5 CONCLUSIONS
Considering the full-state feedback, the performance of Kalman filters design based upon conventional method and singular valued decomposition techniques is compared with observing variations in flow depth and changes in gate opening. No significant difference in performance is observed between both design techniques. Therefore, for irrigation canal, the singular value decomposition (SVD) based Kalman filter is found to be appropriate. In addition, it can be expected that the SVD based Kalman filter formulation for irrigation canals will have highest accuracy and stability characteristics in all existing filter algorithms because SVD is essential for the numerical stability, and is unsurpassed when it comes to producing a numerical solution to a nearly singular system [14]. Another advantage of the SVD based Kalman filter algorithm is the ability to overcome correlated measurement noise. In many present Kalman filter algorithms for irrigation canals, measurement noise are generally designed to be uncorrelated and this may be a large source of error.
Figure 1. Variations in Flow Depth for Conventional and SVD Based Kalman Filter
Figure 3. Changes in Gate Openings for Conventional and SVD-Based Kalman Filter
References:


