Numerical solution of the linear Fredholm-Volterra Integro -Differential equations by the Tau method with an error estimation

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Abstract

The Tau method, produces approximate polynomial solutions of differential, integral and integro-differential equations. in this paper extension of the Tau method has been done for the numerical solution of the general form of linear Fredholm-Volterra Integro-Differential equations. An efficient error estimation for the Tau method is also introduced. Details of the method are presented and some numerical results along with estimated errors are given to clarify the method and its error estimator.

keywords: Tau method; Fredholm and Volterra integral and Integro-Differential equations

1 Introduction

In 1981, Ortiz and Samara [1] proposed an operational technique for the numerical solution of nonlinear ordinary differential equations with some supplementary conditions based on the Tau Method [2]. During the recent years considerable work has been done both in the development of the technique, its theoretical analysis and numerical applications. The same technique has been described in a series of papers [3-7] for the case of linear ordinary differential eigenvalue problems, in [8-12] for the case of partial differential equations and their related eigenvalue problems and in [13] for the iterated solutions of linear operator equations. The object of this paper is to present developments of the operational approach to the Tau Method for the numerical solution of linear Fredholm-Volterra integro-differential equations of the second kind together with mixed supplementary conditions. Special cases of this method solves Fredholm and Volterra integral and integro-differential equations separately. An other special case of this method solves differential equations with mixed supplementary conditions. In [15] Yalcinbas and Sezer, have introduced the Taylor series approximation for the solution of such problems which is a particular case of the Tau method. In this paper, we consider Ortiz and Samara's operational approach to the Tau Method for differential part of the equation, which leads to algorithms of remarkable simplicity, while retaining the accuracy of results.

The organization of this paper is as follows: In section **2**, we introduce the considered problem. In part (**a**) of this section we recall the Tau Method to obtain a matrix form of differential part. In parts (**b**), (**c**) and (**d**), converting other parts of equation to a matrix form is shown. At the end this section corresponding system of linear algebraic equations is given. In Section **3** we recall efficient Tau error estimator. In section **4**, some numerical results are given to clarify the method, where we have computed the numerical results by Maple programming and finally section **5** contains conclusions.

Remark 1.1 *It should be noted that existence and uniqueness of solution of equations is not investigated in this paper.*

2 Fredholm-Volterra integro-differential equations

Consider the following Fredholm-Volterra integro-differential equation together with the given mixed supplementary conditions:

$$Dy(s) - \lambda_1 \int_a^b K_1(s,t)y(t)dt - \lambda_2 \int_a^s K_2(s,t)y(t)dt = f(s), \quad s \in [a,b] \qquad (2.1)$$

$$\sum_{k=1}^{n_d} \left[c_{jk}^{(1)} y^{(k-1)}(a) + c_{jk}^{(2)} y^{(k-1)}(b) + c_{jk}^{(3)} y^{(k-1)}(c)\right] = d_j, \quad j = 1, \cdots, n_d, \qquad (2.2)$$

$$a < c < b.$$

Where D is a linear differential operator of order n_d with polynomial coefficients $p_i(s)$:

$$D = \sum_{i=0}^{n_d} p_i(s) \frac{d^i}{ds^i} \quad ,$$

$$p_i(s) = \sum_{j=0}^{\alpha_i} p_{ij} s^j. \qquad (2.3)$$

If $f(s)$ and $K_i(s,t), (i = 1, 2)$ in (2.1) are not polynomials, they can be approximated by polynomials to any degree of accuracy (by interpolation or Taylor series or other suitable methods).

Unless otherwise stated, s will always be the independent variable of the functions which appear throughout this paper and will be defined in a finite interval. Moreover suppose that $y_n(s)$ be the Tau Method approximation of degree n for $y(s)$, so we can write:

$$p_i(s) = \sum_{j=0}^{\alpha_i} p_{ij} s^j = \underline{p}_i \underline{\underline{s}} \qquad (2.4)$$

$$f(s) = \sum_{j=0}^{n} f_j s^j = \underline{f}\underline{\underline{s}} \qquad (2.5)$$

$$K_i(s,t) = \sum_{p=0}^{n} \sum_{q=0}^{n} k_{pq}^{(i)} s^p t^q, \quad i = 1, 2 \qquad (2.6)$$

$$y_n(s) = \sum_{j=0}^{n} a_j s^j = \underline{a}_n \underline{\underline{s}} \qquad (2.7)$$

where $\underline{p}_i = [p_{i0}, \cdots, p_{i,\alpha_i}, 0, 0, 0, \cdots]$, $\underline{f} = [f_0, \cdots, f_n, 0, 0, 0, \cdots]$,

$\underline{a}_n = [a_0, \cdots, a_n, 0, 0, 0, \cdots]$ and $\underline{\underline{s}} = \left[1, s, s^2, \cdots\right]^T$ are respectively coefficients vectors of $p_i(s)$, right - hand side of equation (2.1), unknown coefficients vector and the basis vector. Without loss of generality we have taken all polynomials of degree n, because if $f(s)$, $K_1(s,t)$, $K_2(s,t)$ and $y_n(s)$ are respectively of different degrees n_f, $(n1_s, n1_t)$, $(n2_s, n2_t)$ and n_y then we can set

$$n = \max\{n_f, n1_s, n1_t, n2_s, n2_t, n_y\}.$$

2.1 Matrix representation for different parts of (2.1)-(2.2)

(a). Matrix representation for $Dy(s)$

The effect of differentiation or shifting (multiplication by the current variable s) on the coefficients $\underline{a}_n = [a_0, a_1, \cdots, a_n, 0, 0, 0, \cdots]$ of a polynomial $y_n(s) = \underline{a}_n \underline{\underline{s}}$ is the same as that of post-multiplication of $\underline{a}_n$ by either the matrix η or the matrix μ:

$$\frac{d}{ds} y_n(s) = \underline{a}_n \eta \underline{\underline{s}}, \quad and \quad s y_n(s) = \underline{a}_n \mu \underline{\underline{s}}$$

where

$$\eta = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 2 & 0 & \\ & & .. & .. \end{bmatrix}, \quad \mu = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & .. & .. \end{bmatrix}.$$

Lemma 2.1 *The effect of r repeated differentiations or q shifts on the coefficients of a polynomial $y_n(s)$ is equivalent to the post-multiplication of $\underline{a}_n$ respectively by η^r or μ^q.*

The proof follows immediately by induction.

Theorem 2.2 *If the operator D and the polynomial $y_n(s)$ are of the forms (2.3), (2.7) then $Dy_n(s) = \underline{a}_n \Pi \underline{\underline{s}}$, where*

$$\Pi = \sum_{i=0}^{n_d} \eta^i p_i(\mu). \qquad (2.8)$$

proof: see [1].

(b). Matrix representation for the Fredholm integral term

It can be seen that

$$\int_a^b K_1(s,t) y_n(t) dt = \underline{a}_n \ \underline{Kf} \ \underline{\underline{s}} \qquad (2.9)$$

where

$$\underline{Kf} = \begin{bmatrix} \sum_{q=0}^n k_{0q}^{(1)} v_{q+1} & \cdots & \sum_{q=0}^n k_{n,q}^{(1)} v_{q+1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ \sum_{q=0}^n k_{0q}^{(1)} v_{q+n+1} & \cdots & \sum_{q=0}^n k_{n,q}^{(1)} v_{q+n+1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \qquad (2.10)$$

and so

$$\underline{Kf}_{lm} = \sum_{i=0}^n k_{mi}^{(1)} v_{i+l+1},$$
$$v_{i+l+1} = \frac{b^{i+l+1} - a^{i+l+1}}{i+l+1}, \qquad l, m = 0, \cdots, n. \qquad (2.11)$$

(c). Matrix representation for the Volterra integral term

Here the columns of the matrix $\underline{Kv}$ associated with Volterra integral term are computed as following:

$$(\underline{Kv})_1 = \begin{bmatrix} -\sum_{q=0}^n \frac{1}{q+1} k_{0q}^{(2)} a^{q+1} \\ -\sum_{q=0}^n \frac{1}{q+2} k_{0q}^{(2)} a^{q+2} \\ \vdots \end{bmatrix} \qquad (2.12)$$

and, for $m = 2, 3, \cdots$

$$
(\underline{Kv})_m =
\begin{bmatrix}
\sum_{q=0}^{m-2} \frac{1}{q+1} k_{m-q-2,q}^{(2)} - \sum_{q=0}^{n} \frac{1}{q+1} k_{m-1,q}^{(2)} a^{q+1} \\
\sum_{q=1}^{m-2} \frac{1}{q+1} k_{m-q-2,q-1}^{(2)} - \sum_{q=0}^{n} \frac{1}{q+2} k_{m-1,q}^{(2)} a^{q+2} \\
\vdots \\
\frac{1}{m-1} k_{00}^{(2)} - \sum_{q=0}^{n} \frac{1}{q+m-1} k_{m-1,q}^{(2)} a^{q+m-1} \\
- \sum_{q=0}^{n} \frac{1}{q+m} k_{m-1,q}^{(2)} a^{q+m} \\
\vdots
\end{bmatrix}. \tag{2.13}
$$

If $a = 0$ then the columns of $\underline{Kv}$ are converted to the following simple forms

$$
(\underline{Kv})_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix},
$$

$$
(\underline{Kv})_m =
\begin{bmatrix}
\sum_{q=0}^{m-2} \frac{1}{q+1} k_{m-q-2,q}^{(2)} \\
\sum_{q=1}^{m-2} \frac{1}{q+1} k_{m-q-2,q-1}^{(2)} \\
\vdots \\
\frac{1}{m-1} k_{00}^{(2)} \\
0 \\
0 \\
\vdots
\end{bmatrix},
$$

$$
m = 2, 3, \cdots . \tag{2.14}
$$

(d). Matrix representation for the supplementary conditions

Replacing $y_n(s)$ from (2.7) into left - hand side of (2.2) we have:

$$
\begin{aligned}
&\sum_{k=1}^{n_d} [c_{jk}^{(1)} y_n^{(k-1)}(a) \\
&+ c_{jk}^{(2)} y_n^{(k-1)}(b) + c_{jk}^{(3)} y_n^{(k-1)}(c)] \\
&= \underline{a}_n B_j
\end{aligned} \tag{2.15}
$$

where $for\ j = 1, \cdots, n_d$,

$$
B_j =
\begin{bmatrix}
\frac{0!}{0!} \left[c_{j1}^{(1)} + c_{j1}^{(2)} + c_{j1}^{(3)} \right] \\
\frac{1!}{1!} \left[c_{j1}^{(1)} a + c_{j1}^{(2)} b + c_{j1}^{(3)} c \right] + \frac{1!}{0!} \left[c_{j2}^{(1)} + c_{j2}^{(2)} \right. \\
\vdots \\
\frac{(n_d-1)!}{(n_d-1)!} \left[c_{j1}^{(1)} a^{n_d-1} + c_{j1}^{(2)} b^{n_d-1} + c_{j1}^{(3)} c^{n_d-1} \right] + \cdots + \frac{(n_d-1)!}{0!} \left[c \right. \\
\frac{n_d!}{n_d!} \left[c_{j1}^{(1)} a^{n_d} + c_{j1}^{(2)} b^{n_d} + c_{j1}^{(3)} c^{n_d} \right] + \cdots + \frac{n_d!}{1!} \left[c_{j,n_d}^{(1)} a \right. \\
\vdots
\end{bmatrix}
$$

We refer to B as the matrix representation of the supplementary conditions and B_j as its j^{th} column.The following relations for computing the elements of the matrix B can be deduced from (2.13):

$$
b_{ij} = \sum_{k=1}^{i} \frac{(i-1)!}{(i-k)!} \left[c_{jk}^{(1)} a^{(i-k)} + c_{jk}^{(2)} b^{(i-k)} + c_{jk}^{(3)} c^{(i-k)} \right],
$$

$$
i, j = 1, \cdots, n_d
$$

and

$$
b_{ij} = \sum_{k=1}^{n_d} \frac{(i-1)!}{(i-k)!} \left[c_{jk}^{(1)} a^{(i-k)} + c_{jk}^{(2)} b^{(i-k)} + c_{jk}^{(3)} c^{(i-k)} \right],
$$

$$
i = n_d + 1, n_d + 2, \cdots, j = 1, \cdots, n_d.
$$

Hence (2.2) can be written as

$$
\underline{a}_n B = \underline{d}, \tag{2.17}
$$

where $\underline{d} = [d_1, \cdots, d_{n_d}]$ is the vector obtained from the right-hand side of (2.2). Consequently, using theorem (2.2) and the results of (**b**), (**c**) and (**d**) parts, we can write the following system of linear equations instead of (2.1) and (2.2):

$$
\begin{cases}
\underline{a}_n (\Pi - \lambda_1 \underline{Kf} - \lambda_2 \underline{Kv}) = \underline{f}, \\
\underline{a} B = \underline{d}.
\end{cases} \tag{2.18}
$$

Now setting

$$\hat{\Pi} = \Pi - \lambda_1 \underline{Kf} - \lambda_2 \underline{Kv} \quad (2.19)$$

$$G_n = \left[B_1, \cdots, B_{n_d}, \hat{\Pi}_1, \cdots, \hat{\Pi}_{n+1-n_d}\right] \quad (2.20)$$

and

$$g_n = [d_1, \cdots, d_{n_d}, f_0, \cdots, f_{n-n_d}], \quad (2.21)$$

(where $\hat{\Pi}_i$ denotes the i^{th} column of $\hat{\Pi}$), the system of equations (2.18) then can be written as

$$\underline{a}_n G_n = g_n, \quad (2.22)$$

which must be solved for the unknown coefficients $a_0, \cdots, a_n$.

Remark 2.3 *For $\lambda_1 = 0$ equation (2.1) is transformed into a Fredholm integro- differential equation and $\lambda_2 = 0$, it is transformed into a Volterra integro- differential equation. For $\lambda_1 = \lambda_2 = 0$, the equation is transformed into a differential equation. For $n_d = 0$ and $p_0(s) = 1$, it is transformed into a Fredholm-Volterra integral equation.*

3 ERROR ESTIMATION OF THE TAU METHOD

In this section an error estimator for the Tau approximate solution of a Fredholm-Volterra integro-differential equation is obtained. Let us call $e_n(s) = y(s) - y_n(s)$ as the error function of the Tau approximation $y_n(s)$ to $y(s)$, where $y(s)$ is the exact solution of (2.1) and (2.2). Hence, $y_n(s)$ satisfies the following problem

$$\begin{aligned} &Dy_n(s) - \lambda_1 \textstyle\int_a^b K_1(s,t)y_n(t)dt \\ &-\lambda_2 \textstyle\int_a^s K_2(s,t)y_n(t)dt = f(s) + H_n(s), \\ &s \in [a,b] \end{aligned} \quad (3.1)$$

$$\begin{aligned} &\textstyle\sum_{k=1}^{n_d}[c_{jk}^{(1)} y_n^{(k-1)}(a) + c_{jk}^{(2)} y_n^{(k-1)}(b) \\ &+c_{jk}^{(3)} y_n^{(k-1)}(c)] = d_j, \\ &j = 1, \cdots, n_d, \end{aligned} \quad (3.2)$$

$$a < c < b.$$

Where $H_n(s)$ is a perturbation term associated with $y_n(s)$ and can be obtained by substituting $y_n(s)$ into the equation

$$\begin{aligned} H_n(s) &= Dy_n(s) - \lambda_1 \textstyle\int_a^b K_1(s,t)y_n(t)dt \\ &-\lambda_2 \textstyle\int_a^s K_2(s,t)y_n(t)dt - f(s). \end{aligned} \quad (3.3)$$

We proceed to find an approximation $e_{n,N}(s)$ to the $e_n(s)$ in the same way as we did before for the solution (2.1) and (2.2).

Subtracting (3.1) and (3.2) from (2.1) and (2.2), respectively, the error function $e_n(s)$ satisfies the equation

$$\begin{aligned} &De_n(s) - \lambda_1 \int_a^b K_1(s,t)e_n(t)dt \\ &-\lambda_2 \int_a^s K_2(s,t)e_n(t)dt = -H_n(s), \\ &s \in [a,b] \end{aligned} \quad (3.4)$$

with the homogeneous conditions

$$\begin{aligned} &\sum_{k=1}^{n_d}[c_{jk}^{(1)} e_n^{(k-1)}(a) + c_{jk}^{(2)} e_n^{(k-1)}(b) \\ &+c_{jk}^{(3)} y_n^{(k-1)}(c)] = 0, \\ &j = 1, \cdots, n_d. \end{aligned} \quad (3.5)$$

Solving this problem in the same way as section (2), we get the approximation $e_{n,N}(s)$(N denotes the Tau degree of $e_n(s)$) . It should be noted that in order to construct the Tau approximation $e_{n,N}(s)$ to $e_n(s)$, only the right-hand side of (2.22) needs to be recomputed, the structure of the coefficient matrix G_n remains the same.

4 NUMERICAL EXAMPLES

In this section, we report on numerical results of some examples, selected through integral and integro-differential equations, solved by the Tau Method described in this paper. We calculated with 15 and 20 digits of accuracy with Maple programming.

For the examples $2-4$, we have reported, in Tables $1-3$, the values of exact solution $y(s)$, Tau approximate solution $y_n(s)$, absolute error $|y(s) - y_n(s)|$ and estimation error (denoted by exact, Tau, Tau-err and Esti.-err, respectively) at selected points of the given interval.

Remark 4.1 *It should be noted that, for the examples 2 and 4 we have used a two variate Taylor expansion to approximate their kernels.*

EXAMPLE 1. ([15],example(1))

$$sy''(s) - sy'(s) + 2y(s) - \int_0^1 (s+t)y(t)dt$$
$$-\int_0^s (s-t)y(t)dt =$$
$$\frac{1}{12}s^4 - \frac{1}{6}s^3 - \frac{1}{2}s^2 - \frac{13}{6}s + \frac{17}{12},$$
$$0 \leq s \leq 1$$
$$y(0) = 1$$
$$y'(0) - 2y(1) + 2y(0) = 1.$$

where $n_d = 2$, $a = 0$, $b = 1$, $p_0(s) = 2$, $p_1(s) = -s$, $p_2(s) = s$, $\lambda_1 = \lambda_2 = 1$, $K_1(s,t) = s+t$, $K_2(s,t) = s-t$ and $f(s) = \frac{1}{12}s^4 - \frac{1}{6}s^3 - \frac{1}{2}s^2 - \frac{13}{6}s + \frac{17}{12}$ Here c can be taken any number between a and b.

We approximate the solution $y(s)$ by the Tau approximate

$$y_2(s) = a_0 + a_1 s + a_2 s^2 \tag{4.1}$$

with $n = 2$.

Then using relations, stated at the parts $(\mathbf{a}), (\mathbf{b}), (\mathbf{c})$ and $(\mathbf{d})$ of section 2, for a $(n+1) \times (n+1)$ matrix, we obtain the following matrices

$$\Pi = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}, K_f = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{3} & 0 \end{bmatrix},$$

$$K_v = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & -2 \end{bmatrix}$$

with

$$\underline{f} = \begin{bmatrix} f_0 & f_1 & f_2 \end{bmatrix} = \begin{bmatrix} \frac{17}{12} & -\frac{13}{6} & -\frac{1}{2} \end{bmatrix},$$

$$\underline{d} = \begin{bmatrix} d_1 & d_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Now using relations $(2.16)-(2.18)$, we find the system of equations (2.19) as follows:

$$\begin{bmatrix} a_0 & a_1 & a_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & -1 & -\frac{1}{3} \\ 0 & -2 & -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1, & 1, & \frac{17}{12} \end{bmatrix}$$

and its solution

$$\underline{a}_2 = \begin{bmatrix} a_0 & a_1 & a_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}.$$

Substituting the elements of this vector into (4.1) we obtain the solution

$$y_2(s) = 1 + s - s^2$$

which is the exact solution.

EXAMPLE 2. ([14], example(b))

$$y'(s) = 1 + 2s - y(s) + \int_0^s s(1+2s)e^{t(s-t)}y(t)dt,$$
$$0 \leq s \leq 1$$
$$y(0) = 1.$$

The exact solution is $y(s) = e^{s^2}$. For numerical results see table 1.

EXAMPLE 3. ([15], example(3))

$$y'(s) + \int_0^s y(t)dt = 1, \qquad 0 \leq s \leq 1$$
$$y(0) = 0.$$

The exact solution is $y(s) = sin(s)$. For numerical results see table 2.

EXAMPLE 4. ([15],example(4))

$$y''(s) + sy'(s) - sy(s) = e^s - 2sin(s)$$
$$+\int_{-1}^1 sin(s)e^{(-t}y(t)dt,$$
$$-1 \leq s \leq 1$$
$$y(0) = 1$$
$$y'(0) = 1.$$

The exact solution is $y(s) = e^s$. For numerical results see table 3.

TABLE 1. Numerical results of Example(2)

$n = 10$				
s	$exact$	Tau	$Tau - err$	$Esti. - err$
0.00	1	1	0	0
.20	1.04081077	1.04081077	$5.72156e - 12$	$5.68889e - 12$
.40	1.17351087	1.17351085	$2.38451e - 08$	$2.33017e - 08$
.60	1.43332941	1.43332623	$3.18608e - 06$	$3.02331e - 06$
.80	1.89648088	1.89637596	$1.04921e - 04$	$9.54437e - 05$
1.00	2.71828183	2.71666667	$1.61516e - 03$	$1.38889e - 03$
$n = 15$				
0.00	1	1	0	0
.20	1.04081077	1.04081077	$1.63300e - 16$	$1.62540e - 16$
.40	1.17351087	1.17351087	$1.08446e - 11$	$1.06522e - 11$
.60	1.43332941	1.43332941	$7.28709e - 09$	$6.99680e - 09$
.80	1.89648088	1.89648013	$7.51118e - 07$	$6.98103e - 07$
1.00	2.71828183	2.71825397	$2.78602e - 05$	$2.48016e - 05$
$n = 20$				
0.00	1	1	0	0
.20	1.04081077	1.04081077	$1.00000e - 19$	$3.60219e - 22$
.40	1.17351087	1.17351087	$4.46000e - 17$	$4.40759e - 17$
.60	1.43332941	1.43332941	$3.39914e - 13$	$3.29740e - 13$
.80	1.89648088	1.89648088	$1.95219e - 10$	$1.84852e - 10$
1.00	2.71828183	2.71828180	$2.73127e - 08$	$2.50521e - 08$

TABLE 2. Numerical results of Example(3)

$n = 5$				
s	$exact$	Tau	$Tau - err$	$Esti. - err$
0.00	0	0	0	0
.20	.19866933	.19866933	$2.60e - 09$	$2.54e - 09$
.40	.38941834	.38941867	$3.24e - 07$	$3.25e - 07$
.60	.56464247	.56464800	$5.53e - 06$	$5.55e - 06$
.80	.71735609	.71739733	$4.12e - 05$	$4.16e - 05$
1.00	.84147098	.84166667	$1.96e - 04$	$1.98e - 04$
$n = 10$				
0.00	0	0	0	0
.20	.19866933	.19866933	$1.00e - 10$	$5.33e - 13$
.40	.38941834	.38941834	$1.00e - 10$	$5.28e - 12$
.60	.56464247	.56464247	$1.00e - 10$	$1.05e - 10$
.80	.71735609	.71735609	$2.20e - 09$	$2.18e - 09$
1.00	.84147098	.84147101	$2.49e - 08$	$2.51e - 08$
$n = 15$				
0.00	0	0	0	0
.20	.19866933	.19866933	$1.00e - 10$	$5.32e - 13$
.40	.38941834	.38941834	$1.00e - 10$	$4.23e - 12$
.60	.56464247	.56464247	0	$1.41e - 11$
.80	.71735609	.71735609	0	$3.30e - 11$
1.00	.84147098	.84147098	0	$6.33e - 11$

TABLE 3. Numerical results of Example(4)

s	$exact$	Tau	$Tau-err$	$Esti.-err$
$n=5$				
-1.00	.36787944	.36657770	$1.30e-03$	$1.28e-03$
$-.80$	.44932896	.44908424	$2.45e-04$	$2.42e-04$
$-.60$	.54881164	.54884907	$3.74e-05$	$3.78e-05$
$-.40$	.67032005	.67037659	$5.65e-05$	$5.67e-05$
$-.20$	.81873075	.81875183	$2.11e-05$	$2.12e-05$
0.00	1	1	0	0
.20	1.22140276	1.22144606	$4.33e-05$	$4.37e-05$
.40	1.49182470	1.49207429	$2.50e-04$	$2.52e-04$
.60	1.82211880	1.82288780	$7.69e-04$	$7.82e-04$
.80	2.22554093	2.22726812	$1.73e-03$	$1.77e-03$
1.00	2.71828183	2.72133477	$3.05e-03$	$3.19e-03$
$n=10$				
-1.00	.36787944	.36787946	$2.29e-08$	$2.27e-08$
$-.80$	.44932896	.44932897	$7.31e-09$	$7.28e-09$
$-.60$	.54881164	.54881164	$2.26e-09$	$2.25e-09$
$-.40$	.67032005	.67032005	$1.43e-10$	$1.39e-10$
$-.20$	.81873075	.81873075	$2.71e-102.71e-10$	
0.00	1	1	0	0
.20	1.22140276	1.22140276	$2e-09$	$1.99e-09$
.40	1.49182470	1.49182468	$2.02e-08$	$2.02e-08$
.60	1.82211880	1.82211868	$1.18e-07$	$1.18e-07$
.80	2.22554093	2.22554039	$5.41e-07$	$5.39e-07$
1.00	2.71828183	2.71827972	$2.11e-06$	$2.10e-06$
$n=15$				
-1.00	.36787944	.36787944	$3.72e-13$	$3.67e-13$
$-.80$	.44932896	.44932896	$2.23e-13$	$2.19e-13$
$-.60$	.54881164	.54881164	$1.00e-13$	$9.75e-14$
$-.40$	.67032005	.67032005	$3.20e-14$	$2.90e-14$
$-.20$	.81873075	.81873075	$3.00e-15$	$3.06e-15$
0.00	1	1	0	0
.20	1.22140276	1.22140276	0	$1.10e-14$
.40	1.49182470	1.49182470	$1.70e-13$	$1.60e-13$
.60	1.82211880	1.82211880	$1.51e-12$	$1.51e-12$
.80	2.22554093	2.22554093	$1.17e-11$	$1.17e-11$
1.00	2.71828183	2.71828183	$7.52e-11$	$7.50e-11$

5 conclusions

Most integro-differential equations are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solutions. For this purpose, the Tau method presented in this paper can be applied. Advantages of the method are that the solution is expressed as a polynomial, the error estimator is available and this method is an exceedingly good approximating method in the sense that its error for problems with reasonable solutions, decays exponentially as the degree of approximation increases [16].

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