Normal splines in reconstruction of multi-dimensional dependencies

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Abstract: The normal spline method is developed for reconstruction of multi-dimensional dependencies in case when information of the function's values and its derivatives is known in nodes of chaotic net in R^n. The function is considered as an element of the Hilbert-Sobolev-Slobodetsky space with fractional differentiation order, and its approximation is an element of minimal norm from the set of interpolation system's solutions. Results of construction a utility function which is rationalized a trade statistics are given.

Key-Words: approximation, generalized splines, normal splines, Hilbert spaces, reproducing kernel, utility function

1 Introduction

The normal spline-collocation (NS) method for linear ordinary differential and integral equations was proposed in [1]. It is effective for singular problems of the given classes [2].

The theoretical basis of the NS method is the classical functional analysis results: the theorem of embedding Sobolev's spaces in the Chebyshev ones [3], and F.Riesz's theorem of canonical representation of linear continuous functionals in Hilbert's spaces as inner products [4]. The last problem is the key one for effective NS algorithms construction.

The NS method consists of some Hilbert-Sobolev norm minimization on the set of a collocation system's solutions. In contrast to the classical collocation methods the basis system here isn't given a priory, but it is constructed according to the chosen norm and to the coefficients of the solving problem. The base functions are canonical images of point wise linear continuous functionals in the chosen space (presented as inner product). To realize this it is necessary to construct the corresponding *reproducing kernel* (RK) defined by the norm [5]. In the case of mentioned above problems of functions of one variable it was sufficient to use classical Sobolev's spaces with integer differentiation order.

In [6] the NS method was developed for two-dimensional problem of computerized tomography. Multivariate generalization of the NS was possible due to usage of the Hilbert-Sobolev-Slobodetsky (HSS) spaces [7]

$$H^d_\varepsilon(R^n) = \left\{\varphi \in S' : (\varepsilon + |\xi|^{d/2}) F[\varphi] \in L_2(R^n)\right\}, \quad (1)$$

were S' - space of the generalized functions of slow growth, $F[\varphi]$ - the Fourier transformation of φ, $\xi \in R^n$, parameter $\varepsilon > 0$, and fractional differentiation order $d > 0$. In the traditional theory $\varepsilon = 1$, however, approximation properties of normal splines are improved at small $\varepsilon > 0$.

Theoretical properties of spaces H^d_ε at $\varepsilon > 0$ are identical. They are Hilbert ones with norm

$$\|\varphi\| = \int\limits_{R^n} \left(\varepsilon + |\xi|^2\right)^d F^2[\varphi]\, d\xi. \quad (2)$$

The corresponding inner product is

$$\langle\varphi, \psi\rangle = \int\limits_{R^n} \left(\varepsilon + |\xi|^2\right)^d F[\varphi]\, \overline{F}[\psi]\, d\xi, \quad (3)$$

where $\varphi, \psi \in H^d_\varepsilon$ and line above $F[\psi]$ means the complex conjunction.

The problem of the getting RK for such kind of spaces has a solution in elementary functions when parameter d is a half-integer value. By this the value $d = (n + 3)/2$ ensures the continuity of functions of the corresponding HSS (such RK is constructed in [6]) and under $d = (n + 5)/2$ such functions are continuously differentiable.

This article is devoted to the problem of multidimensional dependencies interpolation on information of the desired function's values and its gradients in chaotic nodes of some net in the region of the function's factors space. The reproducing kernel for the HSS (1) with $d = (n+5)/2$, the general algorithm of function interpolation and its application to the problem of the utility function that rationalizes some trade statistics [8] are presented.

2 Problem statement and the reproducing kernel

In points $\{x^t : t = \overline{0, T}\}$ of some domain $G \subset R^n$ values u_t of a function $u(x) \in C^2(G)$ and its gradient $q^t \in R^n$ are given. Consider the problem of Hermit's interpolation of this function:

$$u\left(x^t\right) = u_t, \quad \frac{\partial u(x^t)}{\partial x} = q^t, \quad t = \overline{0, T}. \quad (4)$$

This problem also covers the simple interpolation when only the first $T + 1$ equalities are used.

The problem of interpolation is resolved non uniquely, so we precise it on the base of following properties of HSS spaces. It is known [7] that spaces (1) continuously embedded in the Chebyshev ones $C^l(R^n)$ under $d > n/2 + l$. In this case values of function $u(x)$, and also their partial derivatives at the fixed points $x = x^t$ are linear continuous functionals in H^d_ε.

In accordance to the mentioned above Riesz theorem from the theory of Hilbert spaces each of these functionals can be represented as innert product of some element from H^d_ε and $u(x)$. Thus interpolation system (4) is a system of linear equations in H^d_ε. This system is undetermined, but the set of its solutions is convex and closed (as intersection of a finite number of hyperplanes). So, we state the problem to find the solution of system (4) that has a minimal norm (2). It is well known, such solution exists and unique [4]. We named it as *the interpolating normal spline*, and denote as $u^T(x)$.

Consider the problem of canonical representation of linear continuous functionals of the system (4). Such representation should be fulfilled with the help of RK for H^d_ε. On assumption [5], the RK of a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ is such a function $V(\xi, x)$ which possess the following properties:

1) for any $x \in R^n$ the function $V(\cdot, x) \in H^d_\varepsilon$,

2) for any $\varphi \in H^d_\varepsilon$ and $x \in R^n$ the identity

$$\varphi(x) = \langle V(\cdot, x), \varphi\rangle \quad (5)$$

holds.

The last identity means that the function $V(\cdot, x) \in H^d_\varepsilon$ is the canonical presentation (image) of the pointwise functional defined as the value of a function $\varphi(\cdot)$ in a given point x.

In the case of continuous differentiability of the function $V(\cdot, x)$ we can to differentiate the identity (5) and arrive by this to the identity

$$\frac{\partial\varphi(x)}{\partial x_i} = \left\langle \frac{\partial V(\cdot, x)}{\partial x_i}, \varphi \right\rangle \tag{6}$$

which holds for any $\varphi \in H^d_\varepsilon$ and $x \in R^n$. It means, correspondingly, that the function $\partial V(\cdot, x)/\partial x_i$ canonically presents the pointwise functional defined as the value of a function $\partial\varphi(\cdot)/\partial x_i$ in a given point x.

Repeating the technique of [6] one can obtain the RK for H^d_ε under $d = (\ n\ +\ 5\ )/2$:

$$V(\xi, x) = \left(3 + 3\cdot\varepsilon\,|\xi - x| + \varepsilon^2\,|\xi - x|^2\right)\cdot \\ exp\left(-\varepsilon\,|\xi - x|\right). \tag{7}$$

We denote the canonical images of the pointwise functionals (5) for $x = x^t$ as

$$h_t(\xi) = V\left(\xi, x^t\right) = \\ \left(3 + 3\cdot\varepsilon\left|\xi - x^t\right| + \varepsilon^2\left|\xi - x^t\right|^2\right)\cdot \\ exp\left(-\varepsilon\left|\xi - x^t\right|\right), \tag{8}$$

and their derivatives (6) as

$$h'_{ti}(\xi) = \frac{\partial V(\xi, x^t)}{\partial x_i} = \\ -\varepsilon^2(\xi_i - x^t_i)\cdot\left(1 + \varepsilon\left|\xi - x^t\right|\right)\cdot \\ exp\left(-\varepsilon\left|\xi - x^t\right|\ \ \right). \tag{9}$$

We also need in the second derivatives $\partial^2 V(\xi, x)/\partial\xi_i\partial x_j$, but corresponding expressions is omitted in view of their complexities.

3 Algorithm of solving the interpolation problem

After canonical transformation system (4) accepts the form

$$\begin{cases} \langle h_t, u\rangle = u_t, \quad t = \overline{0, T}; \\ \langle h'_{ti}, u\rangle = q^t_i, \quad i = \overline{1, n}. \end{cases} \tag{10}$$

Here the inner product $\langle\cdot,\cdot\rangle$ is defined in accordance to (3). However the described properties of the RK allows to avoid this complicated calculation.

The normal solution of this system, i.e. the normal spline u^T, should be constructed by the generalized Lagrange method [1], [2]. Introduce the Gram matrix of the system (10) functions, i.e. the matrix G with coefficients

$$g_{st} = \langle h_s, h_t\rangle, \quad g_{t\nu(s,i)} = \langle h_t, h'_{si}\rangle, \\ g_{\nu(t,j)\nu(s,i)} = \langle h'_{tj}, h'_{si}\rangle.$$

Here and later index-function $\nu(s, i) = s\cdot n + i$.

Using described properties (5) and (6) of the RK it is easy to receive formulas for elements of matrix G:

$$g_{st} = V(x^s, x^t), \quad g_{s\nu(t,j)} = \frac{\partial V(x^s, x^t)}{\partial x_j}, \\ g_{\nu(s,i)\nu(t,j)} = \frac{\partial^2 V(x^s, x^t)}{\partial\xi_i\partial x_j}. \tag{11}$$

This matrix symmetric and positive definite if all points $\{x^t\}$ are various. It is ensured by linear independence of functions $\{h_t, h'_{ti}\}$ defined by (8) and (9).

By this the realization of the NS method is reduced to formation of the system's Gram matrix (11), to solving the corresponding system of linear algebraic equations

$$\begin{cases} \sum_{s=0}^{T}\left[g_{ts}\lambda_s + \sum_{i=1}^{n} g_{t\nu(s,i)}\mu_{si}\right] = u_t, \quad t = \overline{0, T}; \\ \sum_{s=0}^{t}\left[g_{\nu(t,j)s}\lambda_s + \sum_{i=1}^{n} g_{\nu(t,j)\nu(s,i)}\mu_{si}\right] = q^t_j, \\ \qquad j = \overline{1, n}. \end{cases}$$

with respect to the Lagrange multipliers $\{\lambda_s, \mu_{si}\}$, and to formation of the normal spline

$$u^T(x) = \sum_{s=0}^{T}\left[\lambda_s h_s(x) + \sum_{i=1}^{n}\mu_{si}h'_{si}(x)\right]. \tag{12}$$

4 Numerical construction of a utility function

The problem of construction of an ordinal utility function arises in frame of the inverse problem of the consumer's demand theory [8], [9]. It consists in the construction of such function $u(x)$, depended on vector $x \in R^n_+$ of quantities of bought commodities, that rationalizes an observable demand on some segment of the consumer market. Such demand is presented as a set of values of prices and quantities at observing times $\{p^t, x^t : t = 0, ...T\}$. These expenditure data also determine consumers' costs $b_t = \langle p^t, x^t \rangle$.

In works [10], [11] and others the "non-parametric" method for construction of a utility function has generated. Here on the first stage values of the constructing utility function and corresponding Lagrange multipliers on the observed data $\{u_t = u(x^t),\ \lambda_t = \lambda(p^t, b_t)\}$ are defined as a solution of the Afriat system of linear inequalities. By this gradients of $u(x)$ at $x = x^t$ are $q^t = \lambda_t \cdot p^t$. At the second stage on these outcomes the smooth utility function can be constructed by some interpolation method. The presented above NS method allows to do it on all commodity space R_n^+. Note, that in frame of standard non-parametric method only peace-wise linear utility function had been proposed [11].

The NS method has been tested on some test examples. The expenditure data were formed as follows. A test differentiable utility function $u(x)$ with known analytically demand vector-function $x(p, b)$ was given. Some price-data $\{p^t\}$ and consumers' costs $b_t = const$ were defined. Corresponding quantity-data were calculated as $x^t = x(p^t, b_t)$ and gradients as $q^t = \partial u(x^t) \partial x$. More details see in [9].

Table 1 presents a bank of such data calculated for the case of two commodities and the Thornkvist utility function [8]:

$$u(x_1, x_2) = \frac{x_1^\alpha x_2^{\beta-\alpha}}{(x_1 + \beta + \alpha)^{-\beta}}, \quad \beta > \alpha > 0, \quad (13)$$

where $\beta = 1.5, \quad \alpha = 0.5$. This function presents consumers' preferences when the first commodity is a "necessary" or a cheap one, and another is a "luxury" one.

Table 1: Input data

t	x_1^t	x_2^t	q_1^t	q_2^t
0	0.465	0.977	0.63994	1.27989
1	0.465	0.984	0.63746	1.27981
2	0.464	0.992	0.63495	1.27973
3	0.463	0.999	0.63246	1.27966
4	0.463	1.007	0.62994	1.27958
5	0.462	1.014	0.62745	1.27950
6	0.461	1.006	0.62501	1.27942
7	0.461	0.997	0.62258	1.27934
8	0.460	0.989	0.62012	1.27927
9	0.459	0.981	0.61768	1.27919
10	0.459	0.972	0.61525	1.27911
11	0.458	0.964	0.61279	1.27903
12	0.457	0.956	0.61041	1.27895

The NS method was realized in two variants: simple interpolation (NSI) with use only the first $T + 1$ equalities of (4), and Hermit's one (NSH) with use of all the system. The obtained spline (12) was compared with function (13) which generated the data bank. The comparison was fulfilled on the uniform grid covered the rectangle bounded by extreme values of data $\{x_1^t,\ x_2^t\}$. Corresponding relative error at the grid step 0.01 in the NSI variant equals to 0.055% and in the NSH it equals to 0.071%.

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