Boundary layer flow over a continuously moving wall with suction/injection

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Abstract: - The boundary layer flow past a continuously moving horizontally wall extruding from a slot with velocity $U_w$ into a free-stream moving with velocity $U_\infty$ and subjected to uniform suction/injection, is studied. The boundary layer equations are reduced to the non-similar form and solved using the method developed by Sparrow and Yu [1]. The effect of fluid suction/injection parameter $\xi$ and the parameter $\lambda$, defined as $\lambda = U_\infty / (U_w + U_\infty)$ on the momentum and heat transfer and on the velocity and temperature distributions in the boundary layer are investigated.

Key-Words: - Moving wall; Suction or injection; Skin friction; Heat transfer; Super-velocities

1 Introduction

Studies on heat and mass transfer on boundary layers over continuously moving or stretching surfaces have been on the increase due to their wide range of applications in manufacturing processes such as glass fibre production, metal extrusion, material handling conveyors, extrusion of polymer sheets from die and paper production.

Studies on boundary layer flow past moving surfaces were initiated by Sakiadis [2], who analysed momentum transfer for a flow past a continuously moving plate in an otherwise still fluid. Sakiadis results were later verified experimentally by Tsou et al [3]. Recent studies on boundary layer past moving or stretching surfaces in otherwise quiescent fluids include the work of Ali [4] who looked at similarity solutions for a thermal boundary layer over a power law stretching surface with suction or injection; Elbashbeshy [5] who studied heat transfer over a stretching surface with suction or injection; Magyari and Keller [6] who sought similarity solutions for boundary layer flow over an exponentially stretching surface. Other studies have included the free-stream effects on their boundary layer flows. Abdelhafez [7] and Chappidi and Gunnerson [8] independently considered flows over moving surfaces where both the wall and the free stream moved in the same direction. In their studies they formulated two sets of boundary value problems for the cases $U_\infty < U_w$ and $U_\infty > U_w$. Afzal [9] formulated a single set of equations using as reference velocity, a composite velocity $\bar{U} = U_\infty + U_w$. Later Lin and Huang [10] used Afzal's [9] formulation to study momentum and heat transfer for a flow over a surface moving parallel or reversely to the free stream. A most recent study by Afzal [11] has investigated momentum transfer on a power law stretching surface with free stream pressure gradient.

The current study investigates boundary layer flow past a continuously moving wall extruding from a slot into a moving stream, with suction or injection. The wall is assumed to move in parallel and reversely to free-stream. The governing boundary layer equations are parabolic. The boundary layer flow considered here is generally non-similar except for a few special cases where similarity solutions can be sought. We follow a similar formulation to that used by Afzal [9], where the composite velocity $\bar{U}$ is used as the reference velocity. The non-similar boundary layer equations will be solved using the local non-similar method initiated by Sparrow and co-workers ([1], [12]). This method has been used by quite a number of authors such as
2 Formulation of problem

Consider the flow past a continuously moving infinite wall extruding from a slot with velocity $U_w$ into a free–stream moving with velocity $U_\infty$ and subjected to suction or injection. Assuming that the horizontal wall coincides with the $x$–axis, with $y$–axis being the normal direction to the flow, the dimensional boundary layer equations, upon neglecting viscous dissipation effects and under the Boussinesq approximation are

$$\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\
\frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y} &= \nu \frac{\partial^2 u}{\partial y^2}, \\
\frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} &= \frac{\kappa}{\rho c_p} \frac{\partial^2 T}{\partial y^2},
\end{align*}$$

(1)

where $\rho$ is the fluid’s density, $\nu$ is the kinematic viscosity, $\kappa$ is thermal conductivity and $c_p$ is the coefficient of specific heat of the fluid at constant pressure.

The boundary conditions for this flow are

$$u(x, 0) = U_w, \quad v(x, y) = V_w, \quad T(x, 0) = T_w, \quad u(x, \infty) = U_\infty, \quad T(x, \infty) = T_\infty. \tag{2}$$

Following Afzal et al [9] we let the composite velocity $U_w + U_\infty = \bar{U}$ to be our reference velocity.

The boundary layer equations are non–dimensionalized using the transformations

$$\xi(x) = \frac{V_w}{\bar{U}} Re_x^{1/2}, \quad \eta(x, y) = \frac{y}{x} Re_x^{1/2},$$

$$u(x, y) = \bar{U} f(\xi, \eta), \quad T - T_\infty = (T_w - T_\infty) h(\xi, \eta),$$

where $\xi(x)$ is the boundary layer non–similarity variable and $\eta(x, y)$ is the boundary layer pseudo–similarity variable. Primes here and in the rest of the paper denote differentiation with respect to $\eta$. The parameter $Re_x$ is the local Reynolds number defined as

$$Re_x = \frac{\bar{U} x}{\nu}.$$

Physically $V_w > 0/V_w < 0$ implies fluid is injected/sucked into/from the boundary layer. From the equation of continuity, we have

$$v(x, y) = -\frac{\bar{U}}{2Re_x^{1/2}} \left[ f + \xi (1 + 2\sigma) \frac{\partial f}{\partial \xi} - \eta f' \right],$$

where the parameter $\sigma$ is defined as

$$\sigma(\xi) = \frac{x}{V_w} \frac{dV_w}{dx}.$$

The dimensionless non–similar boundary layer equations take the form

$$f'' = -\frac{1}{2} f f'' + \frac{1}{2} \xi (1 + 2\sigma) \left[ f' \frac{\partial f'}{\partial \xi} - f'' \frac{\partial f}{\partial \xi} \right], \tag{3}$$

$$h'' = \frac{Pr}{2} \left\{ -f h' + \xi (1 + 2\sigma) \left[ f' \frac{\partial h}{\partial \xi} - h' \frac{\partial f}{\partial \xi} \right] \right\}. \tag{4}
$$

The transformed boundary conditions are

$$f(\xi, 0) + (1 + 2\sigma) \frac{\partial f}{\partial \xi}(\xi, 0) = -2\xi,$$

$$f'(\xi, 0) = 1 - \lambda, \quad h(\xi, 0) = 1,$$

$$f'(\xi, \infty) = \lambda, \quad h(\xi, \infty) = 0. \tag{5}$$

The parameters $\lambda$ and $Pr$ are defined respectively as

$$\lambda = \frac{U_\infty}{U_\infty + U_w}, \quad Pr = \frac{\mu c_p}{\kappa}.$$

The case when $\lambda = 1$ (equivalent to $U_w = 0$) corresponds to Blasius flow. The case when $\lambda = 0$ (equivalent to $U_\infty = 0$) represents boundary layer flow over a moving surface in an otherwise still fluid. When $\lambda = 1/2, U_w = U_\infty$ and hence there is no boundary layer.

The wall shear stress and wall heat transfer are represented using the local skin friction factor $c_f$ and the local Nusselt number, $Nu_x$, respectively defined as

$$c_f Re_x^{1/2} = f'(\xi, 0), \quad Nu_x Re_x^{-1/2} = -h'(\xi, 0).$$

Assuming power law variations in fluid suction/injection velocity

$$V_w(x) = v_0 x^n,$$

where $v_0$ is a constant and $\sigma = n$. The flow becomes self–similar if $n = -\frac{1}{2}$. In the current study
the case when \( n = 0 \), corresponding to uniform suction/injection is looked at.

### 2.1 Local non–similarity solutions

We follow closely the method of Sparrow and Yu [1]. The local non–similarity solutions are determined by solving equations (3–5) together with an auxiliary system equations. At the second level of truncation, the auxiliary system is formed by differentiating (3–5) with respect to \( \xi \) and neglecting all terms involving the product \( \frac{\partial^2 (\cdot)}{\partial \xi^2} \). For convenience, a subscript 1 denotes differentiation with respect to \( \xi \); hence for example,

\[
\frac{df_1}{d\xi}, \quad \frac{df_1'}{d\xi}, \quad \frac{dh_1}{d\xi}, \quad \ldots
\]

The auxiliary system takes the form

\[
\begin{align*}
\frac{d^2f_1}{d\xi^2} &= -(1 + n)f_1f''_1 - \frac{1}{2}ff''_1 + \frac{1}{2}(1 + 2n)[f'_1f'_1 + \xi(f'_1)^2 - f''_1f_1], \\
\frac{dh_1}{d\xi} &= \frac{Pr}{2}\left\{ (1 + 2n)n_1f'_1 - 2(1 + n)f_1h_1' - fh_1' + (1 + 2n)[f'h_1 + \xi(f'_1h_1 - h'_1f_1)] \right\},
\end{align*}
\]

with boundary conditions,

\[
\begin{align*}
f_1(\xi, 0) &= 0, \quad f_1(\xi, 0) = -\frac{1}{1 + n}, \quad n \neq -1, \\
h_1(\xi, 0) &= 0, \quad f'_1(\xi, \infty) = 0, \quad h_1(\xi, \infty) = 0.
\end{align*}
\]

Equations (3-4) and (6-7), together with their boundary conditions (5) and (8) can be rewritten as a system of ten first order equations with ten boundary conditions.

### 3 Numerical methods

The local non–similarity solutions are obtained by solving together the boundary layer equations (3) to (5) and the auxiliary system (6) to (8). At any fixed downstream location, \( \xi \), is a constant parameter. The system of equations are hence solved as if they are ordinary differential equations, with the non–similarity variable \( \xi \) being treated as a suction/injection parameter. The system is transformed to a system of ten first order differential equations with ten boundary conditions. The system of first order equations is solved using the fourth–order Runge-Kutta integrator and a Newton–Raphson iterative scheme.

### 4 Discussion of results

The results presented are for \( Pr = 0.7 \). Figures 1 and 2 show the effect of fluid injection on the skin friction and the heat transfer, respectively, as the parameter \( \lambda \) varies. The skin friction coefficient increases with increase in fluid injection except for the case when \( 0.5 < \lambda < \lambda_{cr} \). The increase is more abrupt when \( \lambda > \lambda_{cr} \). The critical value of \( \lambda \) varies with variation in fluid injection. For \( \xi = 0.2, 0.4, 0.5, \lambda_{cr} = 1.06, 0.86, 0.79 \) respectively.

The heat transfer coefficient decreases with increase
in fluid injection for $\lambda < \lambda_{cr}$ but starts to increase abruptly when $\lambda > \lambda_{cr}$. For the case when there is no fluid suction or injection ($\xi = 0$) there is a critical value $\lambda_{cr} = 1.548$ above which no solutions can be obtained. There is also a region close to the $\lambda_{cr}$ where the solutions are not unique. The current results agree exactly with those obtained by Afzal [9]. The skin friction coefficient decreases/increases with increase in fluid suction when $\lambda < 0.5/\lambda > 0.5$, respectively, as shown in Figures 3 and 4. For a fixed $\xi$, $f''(\xi,0)$ increases monotonically with increase in $\lambda$. The heat transfer coefficient increases markedly with increase in fluid suction. For a fixed $\xi$ there is only a slight change in $-h'(\xi,0)$.

Figures 5 and 6 and Figures 7 and 8 show the effect of fluid injection on the velocity and temperature distributions across the boundary layer for the cases when $\lambda = 1.1$ and $\lambda = 0.7$, respectively. For the case when $\lambda = 1.1$, increasing fluid injection results in “super-velocities” in the velocity boundary layer.

Figure 3: Effects of varying fluid suction and $\lambda$ on the skin friction factor.

Figure 4: Effects of varying fluid suction and $\lambda$ on the heat transfer coefficient.

Figure 5: The effect of fluid injection on the velocity distribution in the boundary layer for the case when $\lambda = 1.1$.

Figure 6: The effect of fluid injection on the temperature distribution in the boundary layer for the case when $\lambda = 1.1$.
The effect of fluid injection on the velocity distribution in the boundary layer for the case when \( \lambda = 0.7 \).

Figure 8: The effect of fluid injection on the temperature distribution in the boundary layer for the case when \( \lambda = 0.7 \).

For the case when \( \lambda = 0.7 \), the effect of increasing fluid injection has slight effects on the both the velocity and the thermal boundary layers. The above results and Figures 6 and 7 show clearly that the fluid injection effects are more magnified as \( \lambda \) increases beyond some critical value. For \( \xi = 0.5 \), the super–velocities and the negative temperatures occur when \( \lambda > 0.8 \) and their peaks intensify with increases in \( \lambda \).

5 Conclusion

We have looked at the effect of fluid suction/injection on the boundary layer flow of a moving infinite wall extruding from a slot into a free–stream. Increasing fluid injection results in increase in skin friction and decrease in the heat transfer coefficient.
for \( \lambda < \lambda_{cr} \). Both the skin friction factor and the heat transfer coefficient increase abruptly with increase in fluid injection when \( \lambda > \lambda_{cr} \). The skin friction factor decreases/increases with increase in fluid suction when \( \lambda < 0.5/\lambda > 0.5 \), respectively. The heat transfer increases quite fast with increase in fluid suction. For \( \lambda > \lambda_{cr} \) increases fluid injection results in “super–velocities” in the velocity boundary layer and negative temperature in the thermal boundary layer. The appearance of “super–velocities” is probably an indication that the fluid injection acts to accelerate fluid in the downstream direction. The appearance of negative temperature within the thermal boundary layer is a novel feature which may be could mean that when \( \lambda > \lambda_{cr} \), fluid injection greatly enhances heat transfer not only from the wall but also from the boundary layer.

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References: