Freedom in the Normal Form Expansion and Obstacles to Asymptotic Integrability: The Perturbed KdV Equation

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Abstract: Integrable evolution equations, to which a perturbation is added, often lose their integrability when multiple-wave solutions are considered. In the Normal Form expansion, this leads to the emergence of terms that the formalism cannot account for. In the standard application of the method, these terms spoil the integrability of the normal form (hence, “obstacles to asymptotic integrability”), leading to distortion of the multiple-wave structure of the zero-order approximation, even asymptotically far from the interaction region of the waves. While obstacles do not emerge in the case of single-wave solutions, this is not borne out by the analysis of the general case. Exploiting the freedom inherent in the Normal Form method, an alternative expansion algorithm is proposed, resulting in obstacles that are expressed in terms of symmetries of the unperturbed equation, and which vanish explicitly for single-wave zero-order solutions. The normal form remains integrable, generating multiple-wave solutions of the same structure as generated by the unperturbed equation, because the effect of the obstacles is shifted from the normal form to the higher-order terms in the expansion of the solution. The obstacles affect the solution only in the interaction region of the waves, which is a finite domain around the origin in the $x$-$t$ plane in the case of solitons. Results are presented for the perturbed KdV equation.

Key-Words: Obstacles to integrability; Normal forms; Freedom in expansion

1 Introduction

Integrable evolution equations, to which a small nonlinear perturbation is added, often lose their integrability for solutions that evolve from a multiple-wave unperturbed solution [1-10]. Methods, such as inverse scattering, Lie symmetry analysis and the Normal Form expansion fail to produce a closed form expression for the solution, or for its perturbative expansion. Here, we address this issue within the framework of the Normal Form method, where this phenomenon manifests itself in the form of obstacles to asymptotic integrability - terms that the formalism cannot account for, which spoil the integrability of the normal form. As a result, when a multiple-wave zero-order approximation (solution of the normal form) is sought, its structure is distorted even asymptotically far from the origin. In the direct analysis of solutions that evolve from a single-wave unperturbed solution, such obstacles do not emerge [2,10]. The analysis of the general case does not reproduce this result.

In this paper we briefly report how exploitation of the freedom inherent in the Normal Form method, allows avoidance of these difficulties. The results [11] for the case of the perturbed KdV equation are presented. We have also analyzed the perturbed Burgers and heat diffusion equations [12,13], and the algorithm presented below yields similar results. The normal form is left intact, so that it is integrable, generating multiple-wave solutions of the same structure as obtained for the unperturbed equation. The obstacles obtain a “canonical” form, expressed in terms of symmetries of the unperturbed equation, and vanish explicitly in the case of a single-wave zero-order solution. Their effect is shifted to the corrections to the solution in the order in which they appear. It is confined to the interaction region of the waves. Away from that region, the canonical obstacles decay exponentially to zero. In the case of solitons, this region is a finite domain around the origin in $x$-$t$ plane. As a result, the obstacles do not generate secular terms. In the case of fronts, (as in the perturbed Burgers equation) the interaction region may be an infinite line in the $x$-$t$ plane. Then the asymptotic effect of the obstacles
can often be found in closed form, indicating the formation of a new wave-front [13].

2 Standard Normal Form Expansion

Consider an integrable equation

$$\partial_t w = F_0[w]$$  (1)

$F_0[w]$ is a differential polynomial in $w$. For instance, in the case of the KdV equation,

$$F_0[w] = 6ww_t + w_{xxx}$$  (2)

Eq. (1) is assumed to have single- and multiple-wave solutions. In the case of the KdV equation, the single-soliton solution is given by [14-17]

$$u(t,x) = 2k^2/cosh^2(k(x + vt + x_0))$$  (3)

A simple dispersion relation relates the wave vector, $k$, and the velocity, $v$:

$$v = 4k^2$$  (4)

Multi-soliton solutions are given by the Hirota algorithm [18]. For the two-soliton case it yields

$$u(t,x) = 2\partial_t^2 \ln \left\{ \frac{1 + g_1(t,x) + g_2(t,x) + 2 k_1 k_2 g_1(t,x) g_2(t,x)}{k_1 - k_2} \right\}$$  (5)

$$g_i(t,x) = \exp[2k_i(x + vt + x_{0,i})]$$

Each pair, $(k_i, v_i)$ obeys Eq. (4). When $|x|, |t| \to \infty$, the solution breaks up into two separated solitons:

$$u(t,x) \to \frac{2k_1^2}{cosh^2(k_1(x + v_1 t + x_{0,1}))} + \frac{2k_2^2}{cosh^2(k_2(x + v_2 t + x_{0,2}))}$$  (6)

It loses its simple two-soliton character in the interaction region, which is a finite domain around the origin in the $x$-$t$ plane.

Consider now a perturbed version of Eq. (1):

$$\partial_t w = F_0[w] + \varepsilon F_1[w] + \varepsilon^2 F_2[w] + O(\varepsilon^3)$$  (7)

In the case of the perturbed KdV equation, the perturbation terms are customarily written as

$$F_1[w] = 30\alpha_1 w^2 w_x + 10\alpha_2 ww_{xxx} + 20\alpha_3 w_x w_{xx} + \alpha_4 w_{xxx} +$$  (8)

$$F_2[w] = 140\beta_1 w^3 w_x + 70\beta_2 w^2 w_{xxx} + 280\beta_3 w_x w_{xx} + 14\beta_4 w_x w_{xxx} +$$  (9)

$$70\beta_5 w^3 + 42\beta_6 w_x w_{4x} + 70\beta_7 w_x w_{xx} + \beta_8 w_x$$

The Normal Form analysis involves two asymptotic expansions: (i) The expansion of the solution in a power series in $\varepsilon$:

$$w(t,x) = u(t,x) + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + O(\varepsilon^3)$$  (10)

also called the Near Identity Transformation (NIT); (ii) The normal form, which yields the time dependence of $u$, the zero-order approximation:

$$u_t = U_0[u] + \varepsilon U_1[u] + \varepsilon^2 U_2[u] + O(\varepsilon^3)$$  (11)

Substituting Eqs. (10) and (11) in Eq. (7), one solves the problem order-by-order in powers of $\varepsilon$. In every order one obtains a homological equation. The first-order homological equation reads

$$U_1 + \partial_t u^{(1)} = [F_0[u], u^{(1)}] + F_1[u]$$  (12)
The expression in square brackets is called the Lie brackets of $F_0$ and $u^{(1)}$. For two functions, $M$ and $N$, which depend on $t$ and $x$ either explicitly, or through their dependence on $u$ and its spatial derivatives, the Lie brackets are defined as

$$[M,N] = \sum_i \left( \frac{\partial M}{\partial u_i} \partial_x^i N - \frac{\partial N}{\partial u_i} \partial_x^i M \right)$$

(13)

The $n$th-order homological equation cannot determine both $U_n$ and $u^{(n)}$ (see, e.g., Eq. (12)). $U_n$ are determined by assigning to them all “resonant” terms (symmetries of the unperturbed equation), generated by the perturbation (e.g., $F_1[u]$ in Eq. (11)). This ensures that $u^{(n)}$ do not contain secular terms. It is expected that the remaining, non-resonant, terms will be accounted for by $u^{(n)}$.

In the case of the KdV equation, the first three terms in the normal form are given by:

$$U_0[u] = S_2[u]$$

$$U_1[u] = \alpha_4 S_3[u], \quad U_2[u] = \beta_8 S_4[u]$$

(14)

$S_n$, the symmetries of the unperturbed equation [19-21], are differential polynomials in $u$ whose Lie-Brackets with $F_0$ as well as with one another vanish:

$$[S_n[u], S_m[u]] = 0$$

(15)

The first four symmetries are given below:

$$S_1 = u_x, \quad S_2 = F_0[u] = 6 u u_x + u_{xxx}$$

$$S_3 = 30 u^2 u_x + 10 uu_{xxx}$$

$$+ 20 uu_{x} u_x + u_{5x}$$

$$S_4 = 140 u^3 u_x + 70 uu_{xx}$$

$$+ 280 uu_{x} u_x + 14 uu_{xx}$$

$$+ 70 u_x^3 + 42 uu_{4x}$$

$$+ 70 uu_{x} uu_{xx} + u_{7x}$$

(16)

There is more freedom in the normal form expansion. Intuitively, $u^{(n)}$ of Eq. (10) are expected to be differential polynomials in $u$. In addition, there may be an explicit dependence of $u^{(n)}$ on $t$ and $x$. In the standard analysis [1-10], it is assumed that such explicit dependence does not exist.

2.1 No obstacle in $O(\epsilon)$

With this assumption, Eq. (12), becomes

$$U_1 = [F_0[u], u^{(1)}] + F_1[u]$$

(17)

With $U_1$ given in Eq. (14), $F_1[u]$ in Eq. (17) is fully accounted for by $u^{(1)}$, which is a differential polynomial in $u$ [2,5,6,10]:

$$u^{(1)}[u] = (-5 \alpha_1 + 4 \alpha_3 + 20 \alpha_4) u^2$$

$$+ \left( -\frac{10}{3} \alpha_3 + \frac{10}{3} \alpha_4 \right) u u_x$$

$$+ \left( -\frac{5}{3} \alpha_1 + \frac{5}{3} \alpha_2 + 5 \alpha_4 \right) u_x$$

(18)

$$q^{(1)} = \int u(x,t) dx$$

Hence, no obstacle emerges in $O(\epsilon)$.

2.2 Second-order obstacle

The $O(\epsilon^2)$ homological equation has the form

$$U_2 = \partial_u u^{(2)} = [F_0[u], u^{(1)}] + F_2[u] + Z^{(2)}[u]$$

(19)

Here $Z^{(2)}$ is the contribution of lower-order terms. In the standard analysis one assumes that $u^{(2)}$ has no explicit $t$- and $x$- dependence:

$$u^{(2)} = u^{(2)}[u]$$

(20)

Eq. (19), therefore, becomes

$$U_2 = [F_0[u], u^{(1)}] + F_2[u] + Z^{(2)}[u]$$

(21)

The formalism leads to an impasse for a variety of perturbed equations [1-10]. In the case of the KdV equation it allows $u^{(2)}$ to have the following structure:

$$u^{(2)} = Au^3 + Bu^2 q^{(1)} + Cu u_x q^{(1)}$$

$$+ Du u_{xx} + Eu q^{(1)} + Fu q^{(1)}$$

$$+ Gu^2 + Hu u_x q^{(1)} + I u_x q^{(2)}$$

$$+ Ju_x q^{(1)} + Ku u_{xx} q^{(1)} + Lu_{xxx}$$

(22)
In Eq. (22), $q^{(2)} = \partial_x^{-1}(u^2)$. With $U_2$ given in Eq. (14), accounting for as many terms as possible in $F_2 + Z^{(2)}$ of Eq. (21), one finds that $B = E = F = H = 0$. The remaining coefficients are calculable in terms of the $\alpha$’s and $\beta$’s of Eqs. (8) and (9). Still, one contribution on the r.h.s. of Eq. (21),

$$R^{(2)}_{St} = \mu u^3 u_x$$

remains unaccounted for. In Eq. (23), $\mu$ is a known polynomial in the $\alpha$ and $\beta$ coefficients of Eqs. (8) and (9). The subscript $St$ stands for the “standard” analysis. Fig. 2 shows an example of this obstacle in the two-soliton case.

As $R^{(2)}_{St}$ cannot be accounted for in Eq. (21) by the contribution of $u^{(2)}$ to the Lie bracket, it must be included in $U_2$:

$$U_2 = \beta_s S_4[u] + R^{(2)}_{St}$$

(24)

As a result, the normal form becomes

$$u_s = S_2[u] + \varepsilon \alpha_s S_1[u] + \varepsilon^2 \left( \beta_s S_4[u] + R^{(2)}_{St} \right) + O(\varepsilon^3)$$

(25)

A normal form without obstacles generates the same multiple-soliton solutions as the unperturbed KdV equation, with Eq. (4) replaced by [22-24]

$$v = 4 k^2 - 16 \varepsilon \alpha_s k^4 + 64 \varepsilon^2 \beta_s k^6 + O(\varepsilon^3)$$

(26)

The zero- and first-order terms in Eq. (25) are symmetries, which guarantee integrability through $O(\varepsilon)$. $R^{(2)}_{St}$ is an obstacle to asymptotic integrability in $O(\varepsilon^3)$. It generates an inelastic effect: the velocities of the outgoing solitons differ from those of the incoming ones in $O(\varepsilon^4)$ [2,10].

2.3 Single-soliton case

Repeating the analysis for a single-soliton zero-order solution (Eq. (3) with Eq. (25)), obstacles do not emerge [2,10]. The normal form is integrable, as it is constructed out of symmetries only. However, the obstacle $R^{(2)}_{St}$ of Eq. (22), which has been found in the analysis of the general case, does not vanish if one substitutes the single-soliton solution for $u$. Technically, this apparent discrepancy is a consequence of the fact that some of the algebraic steps carried out in the general analysis are not possible in the single-soliton case.

3 Freedom in Expansion & Obstacles

The freedom inherent in the expansion procedure may be exploited to partially overcome the difficulties encountered owing to the loss of integrability. This is achieved in three steps.

Step 1 One performs the perturbative analysis of Eq. (7) with $u$ assumed to be a single-soliton, given by Eq. (3), but with $k$ and $v$ obeying Eq. (26). No obstacles are encountered. Three of the coefficients in the expression for $u^{(2)}$, Eq. (22), say, $A$, $D$, and $J$, are determined in terms of $B$, $C$, $E$, $F$, $G$, $H$, $I$, $K$, and $L$, which are free, and the $\alpha$’s $\beta$’s of Eqs. (8) and (9). We denote the resulting differential polynomial by $u_\varepsilon^{(2)}[u]$.

Step 2 Returning to the analysis of the general case, one assumes:

$$u^{(2)} = u^{(2)}_s$$

(27)

However, now $u^{(2)}_s[u]$ is computed for $u$ that solves the normal form in the general case. One encounters an obstacle. Its general form can be written in terms of “canonical” obstacles:

$$R^{(2)} = Q_1 u R^{(2)}_{12} + Q_2 \partial_x R^{(2)}_{12} + Q_3 u_\varepsilon \partial_x^{-1} R^{(2)}_{12} + Q_4 R^{(2)}_{13}$$

(28)
The coefficients \( Q_i \) (1 \( \leq i \leq 4 \)) are known combinations of the coefficients of differential monomials in \( u^{(2)} \) as well as the \( \alpha 's \) and \( \beta 's \). The terms \( R_{nm} \) are expressed in terms of the symmetries of the unperturbed equation:

\[
R_{nm} = G_n[u] S_m[u] - G_m[u] S_n[u]
\]

(29)

\( R_{nm} \) have two important properties. First, they vanish identically when \( u \) is single-soliton solution is substituted for \( u \). This is a consequence of the fact that in the latter case all the symmetries are proportional to \( S_i \):

\[
S_n[u] = (-v)^n S_i = (-v)^n u_i
\]

(30)

Eq. (30) can be proven by induction, using the known recursion relation that \( S_n \) obey [19-21]. Second, in the multi-soliton case, \( R_{nm} \) decay exponentially fast away from the interaction region of the individual solitons. The reason is that, away from this region (a finite domain around the origin in the \( x-t \) plane), except for exponentially small corrections, the multi-soliton solution breaks up into a sum of independent single-solitons. As \( R_{nm} \) vanish identically for single solitons, only the exponentially small corrections remain. An example of \( u-R_{21} \) for a two-soliton zero-order solution is given in Fig. 3. The dip in the figure corresponds to a negative peak identical in shape to the visible peak.

\[
R^{(2)} = \frac{9}{10} \mu R_{21}
\]

(31)

In Eq. (31), \( \mu \) is the same coefficient as in Eq. (23).

Returning now to Eq. (25) and replacing \( R_{nk}^{(2)} \), the (non-localized) obstacle obtained in the standard analysis, by the localized obstacle of Eq. (31), the effect on the solution of the normal form is confined to the soliton-interaction region. Away from it, the equation is reduced to one comprised of symmetries only.

Step 3 As the obstacle is localized in the \( x-t \) plane, it pays to transfer its burden in the homological equation, Eq. (19), from the normal form to the NIT. This is achieved by allowing \( u^{(2)} \) to have explicit dependence on \( x \) and \( t \) and modifying Eq. (27) into

\[
u^{(2)} = u^{(2)}_i [u] + u^{(2)}_r (t, x)
\]

(32)

In Eq. (32), \( u^{(2)} \) is an unknown term, explicitly dependent on \( t \) and \( x \). Substituting Eq. (32) in Eq. (19), and requiring that \( U_2 \) assumes its value given in Eq. (14), Eq. (19) becomes an equation for \( u^{(2)}_r \):

\[
\partial_x u^{(2)}_r (t, x) = 6 \partial_x \left\{ uu^{(2)}_r (t, x) \right\} + \partial_x^3 u^{(2)}_r (t, x) + \frac{9}{10} \mu R_{21}
\]

(33)

As the obstacle is confined to a small region around the origin, it does not generate a secular term in \( u^{(2)} \). The normal form is, again, constructed from symmetries only, and is, hence, integrable, generating the standard multi-soliton solutions.

References:


Veksler A. and Zarmi Y., Overcoming obstacles to integrability in perturbed nonlinear evolution equations, to be published, 2004


