Weighted Mean Matrix on Weighted Sequence Spaces

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Abstract: In a recent paper [8], the author has discovered the norm for the Cesaro, Copson and Hilbert operators on Lorentz sequence space $d(w,1)$. The purpose of this note is to establish analogous norms for arbitrary weighted mean matrices (with non-negative entries) acting on arbitrary $\ell_1(w)(d(w,1))$ spaces.

Key Words: Norm, Weighted Mean Matrix, Weighted Sequence Space.

1. Introduction

The purpose of this paper is the problem of finding the norm of weighted mean operator, when $\ell_p$ [5] is replaced by the weighted $\ell_1$-space $\ell_1(w)$ and the Lorentz sequence space $d(w,1)$, determined by a weighting sequence $w = (w_n)$.

We prove here two main results, the first is finding the norm of weighted mean operator, where the weighted mean matrix, $A_d$, is an infinite matrix of the form

$$a_{n,k} = \begin{cases} \frac{d_n}{D_n} & \text{for } 1 \leq k \leq n \\ 0 & \text{for } k > n \end{cases}$$

and the $d_n$’s are non-negative numbers with partial sum $D_n = d_1 + \ldots + d_n$ (we insist that $d_1 > 0$, so that each $D_n$ is positive.) such matrices arise naturally in summability theory and have been studied extensively from this point of view (see, for example, [3]. These results are extensions of some results in [7] and [8], where the author solved this problem for the averaging operator and its transpose (Copson operator).

Let the operator have matrix $A_d = (a_{n,k})$, defined as above, and let $v_n = \sum_{k=1}^{n} a_{k,n} w_k$, that is $v_n = \sum_{k=1}^{\infty} \frac{w_k d_k}{D_n}$. Write $V_n = v_1 + \ldots + v_n$ (and $W_n$ similarly). Then the norm of $A_d$ is the supremum on $\frac{V_n}{W_n}$. This amount to saying that is determined by elements of the form $(1, \ldots, 1, 0, \ldots)$.

Typical results are as follows. For $w_n = \left( \frac{1}{n^p} \right)$, where $0 < p \leq 1$, the weighted mean matrix $B_d$, where $d_n = \left( \frac{1}{n^p} \right)$ such that $0 < p - q \leq 1$, has norm $\zeta(p-q+1)$. The averaging operator has norm $\zeta(p+1)$.

The case of $d(w,p)$ with $p > 1$ presents substantial additional features. Some results for this case are given in [6] and [7].

1. General Weighted Mean Matrix Operator

We start by describing our problem in the notation of weighted $\ell_1$-spaces. Let $w = (w_n)$ be a decreasing, non-negative sequence with $\lim_{n \to \infty} w_n = 0$ and $\sum_{n=1}^{\infty} w_n$ divergent. Write $W_n = w_1 + \ldots + w_n$. Then $\ell_1(w)$ (and the Lorentz sequence space $d(w,1)$) is the space of sequences $x = (x_n)$ with

$$\|x\|_{\ell_1(w)} = \sum_{n=1}^{\infty} w_n |x_n| \quad \& \quad \|x\|_{w,1} = \sum_{n=1}^{\infty} w_n x_n^*,$$

convergent, where $(x_n^*)$ is the decreasing rearrangement of $|x_n|$.

We now consider the operator $B$ defined by $Bx = y$, where $y_n = \sum_{k=1}^{\infty} b_{n,k} x_k$. We shall write $\|B\|_{\ell_1(w)}$ for the norm of $B$ when regarded as an operator from $\ell_1(w)$ to $\ell_1(w)$, where

$$\|B\|_{\ell_1(w)} = \sup \{ \|Bx\|_{\ell_1(w)} : \|x\|_{\ell_1(w)} \leq 1 \},$$

$$\|B\|_{w,1} = \sup \{ \|Bx\|_{w,1} : \|x\|_{w,1} \leq 1 \}.$$
Also, we define

\[ M_{w,1}(B) = \sup \{ \| Bx \|_{\ell_1(w)} : \| x \|_{\ell_1(w)} = 1 \} , \]

where \( x = (x_n) \) is regarded as a decreasing, non-negative sequences in \( \ell_1(w) \).

We assume that

1) \( b_{n,k} \geq 0 \) for all \( n, k \). This implies that \( |Bx| \leq B(|x|) \) for all \( x \), and hence the non-negative sequences \( x \) are sufficient to determine \( \| B \|_{\ell_1(w)} \).

We assume further that each \( B(c_k) \) is in \( \ell_1(w) \), that is:

2) \( \sum_{n=1}^{\infty} w_n b_{n,k} \) is convergent for each \( k \), that guarantee each \( B(c_k) \) is in \( \ell_1(w) \).

For two finite sequence \( x = (x_n) \) and \( y = (y_n) \), write \( y \ll x \) if

\[ Y_k \leq X_k \quad (\forall k), \]

where \( X_k = \sum_{i=1}^{k} x_i \) and \( Y_k = \sum_{i=1}^{k} y_i \).

**Lemma 1:** Suppose \( x, y \in \mathbb{R}^n \) with \( x \ll y \) or \( (a_i) \) is decreasing. Suppose also either \( a_n \geq 0 \), or \( X_n = Y_n \).

Then

\[ \sum_{k=1}^{n} a_k y_k \leq \sum_{k=1}^{n} a_k y_k. \]

**Proof:** By Abel summation, it follows that

\[ \sum_{k=1}^{n} a_k y_k = \sum_{k=1}^{n} a_k (Y_k - Y_{k-1}) \quad (Y_0 = 0) \]

\[ = \sum_{k=1}^{n} Y_k (a_k - a_{k-1}) + a_n Y_n. \]

Now applying the hypothesis in both cases, we deduce that

\[ \sum_{k=1}^{n-1} Y_k (a_k - a_{k+1}) + a_n Y_n \geq \sum_{k=1}^{n-1} X_k (a_k - a_{k+1}) \]

\[ + a_n X_n \]

\[ = \sum_{k=1}^{n} a_k x_k. \]

Therefore

\[ \sum_{k=1}^{n} a_k x_k \leq \sum_{k=1}^{n} a_k y_k. \]

**Corollary:** Let \( x, y \) be decreasing, non-negative elements of \( \mathbb{R}^n \) (or \( \ell_1(w) \)) with \( x \ll y \). Then

\[ \| x \|_{\ell_1(w)} \leq \| y \|_{\ell_1(w)}. \]

**Proposition 1:** Suppose that (1) holds, and that

(3) for all subsets \( M, N \) of \( I \) having \( m, n \), elements respectively, we have \( \sum_{i \in M} \sum_{j \in N} b_{i,j} \leq \sum_{i=1}^{m} \sum_{j=1}^{n} b_{i,j} \).

Then \( \| Bx \|_{\ell_1(w)} \leq \| Bx^* \|_{\ell_1(w)} \) for all non-negative elements \( x \) of \( \ell_1(w) \) (with \( d(w,1) \)), where \( x^* \) is the decreasing rearrangement of \( |x_n| \). Hence decreasing, non-negative sequence \( x \) are sufficient to determine \( \| B \|_{\ell_1(w)} \).

**Proof:** This proof, which is not given here, stated in [8], Proposition 1. **Proposition 2** (2, Lemma 9): Let \( B = (b_{i,j})_{i,j=1}^{\infty} \) be a matrix operator with non-negative entries, and consider the associated transformation, \( x \rightarrow y \), given by \( y_i = \sum_{j=1}^{\infty} b_{i,j} x_j \). Then the following conditions are equivalent:

(i) \( y_1 \geq y_2 \geq \ldots \geq 0 \) whenever \( x_1 \geq x_2 \geq \ldots \geq 0 \).

(ii) \( r_{i,n} \geq r_{i+1,n} \quad (i, n = 1, 2, \ldots) \), where \( r_{i,n} = \sum_{j=1}^{n} b_{i,j} \).

**Proof:** (i) \( \Rightarrow \) (ii) follows by taking \( x \) to be the sequence \((1, 1, 1, 0, \ldots)\) of \( n \) ones followed by zeros.

(ii) \( \Rightarrow \) (i): By Abel summation, it follows that

\[ y_i = \sum_{j=1}^{\infty} b_{i,j} x_j = \sum_{n=1}^{\infty} r_{i,n} (x_n - x_{n+1}). \]

Since \( r_{i,n} \geq r_{i+1,n} \forall i, n \), and also \((x_n)\) is decreasing, non-negative sequence, then

\[ \sum_{n=1}^{\infty} r_{i,n} (x_n - x_{n+1}) \geq \sum_{n=1}^{\infty} r_{(i+1),n} (x_n - x_{n+1}) \]

\[ = \sum_{j=1}^{\infty} b_{i+1,j} x_j = y_{i+1}. \]

This completes the proof of the statement.

Let \( A_d \) be the weighted mean matrix operator that defined as follows [1]:

\[ (A_d x)(n) = \frac{1}{D_n} (d_1 x_1 + d_2 x_2 + \ldots + d_n x_n), \]

where \( \{d_n\} \) is the non-negative, decreasing sequence with \( d_n > 0 \), and \( D_n = d_1 + \ldots + d_n \).

**Proposition 3:** If \( B \) satisfies conditions (1), (2) and (3), then

\[ \| B \|_{w,1} = M_{w,1}(B). \]
**Proof:** By Proposition 1, \( \|B\|_{\ell_1(w)} \) is determined by decreasing, non-negative sequence \( x \). Since \( r_{i,n} \) decreases with \( i \) for each \( n \), if \( x \) is decreasing and non-negative, then so \( Bx \), so that \( \|B\|_{\ell_1(w)} = \|B\|_{\ell_1(w)} \).

**Theorem 1:** Suppose \( B_d \) is a weighted mean operator on \( \ell_1(w) \). If

\[
\sup_n \frac{U_n}{W_n} < \infty,
\]

where \( U_n = u_1 + \ldots + u_n \) and \( u_n = \sum_{k=n}^{\infty} \frac{w_k}{d_k} \), and \( W_n = w_1 + \ldots + w_n \), then \( B_d \) is a bounded operator from \( \ell_1(w) \) into itself, and also

\[
M_{\ell_1(w)}(B_d) = \sup_n \frac{U_n}{W_n} = \|B_d\|_{\ell_1(w)}.
\]

**Proof:** Since \( (B_dx)(n) \leq (B_dx^*)(n) \) for all \( n \), it is sufficient to consider decreasing, non-negative sequences. Let \( x \) in \( \ell_1(w) \) be such that \( x_1 \geq x_2 \geq \ldots \geq 0 \). Then

\[
\|B_dx\|_{\ell_1(w)} = \|B_dx\|_{\ell_1(w)} = \sum_{n=1}^{\infty} w_n \left( \frac{1}{D_n} \sum_{k=1}^{n} d_k x_k \right)
\]

\[
= \sum_{n=1}^{\infty} u_n x_n
\]

\[
= \sum_{n=1}^{\infty} U_n (x_n - x_{n+1}).
\]

Also, we have

\[
\|x\|_{\ell_1(w)} = \sum_{n=1}^{\infty} w_n x_n = \sum_{n=1}^{\infty} W_n (x_n - x_{n+1}).
\]

Let \( U = \sup_n \frac{U_n}{W_n} \). Then

\[
\|B_dx\|_{\ell_1(w)} \leq U \sum_{n=1}^{\infty} W_n (x_n - x_{n+1})
\]

\[
= \sum_{n=1}^{\infty} w_n x_n
\]

\[
= U \|x\|_{\ell_1(w)} = U \|x\|_{\ell_1(w)}.
\]

Hence \( M_{\ell_1(w)}(B_d) \leq U \).

To show that the constant is the best possible, we take \( x_1 = x_2 = \ldots = x_n = 1 \), and \( x_k = 0 \) for all \( k \geq n + 1 \). Then

\[
\|x\|_{\ell_1(w)} = W_n \quad \& \quad \|B_d\|_{\ell_1(w)} = U_n.
\]

Therefore

\[
M_{\ell_1(w)}(B_d) = U = \|B_d\|_{\ell_1(w)}.
\]

**Proposition 4:** Suppose that \( B \) satisfies conditions (1) and (2). Let \( u_j = \sum_{i=1}^{\infty} b_{i,j} w_i \) and \( U_n \) as usual. Then

\[
\|B\|_{\ell_1(w)} = \sup_{n \geq 1} \frac{u_n}{w_n}.
\]

**Proof:** Let \( \sup_{n \geq 1} \frac{u_n}{w_n} = M \). If \( x = e_j \), then \( \|x\|_{\ell_1(w)} = w_j \), while \( \|Bx\|_{\ell_1(w)} = u_j \). Hence \( \|B\|_{\ell_1(w)} \geq M \) (also when \( M = \infty \)). Now suppose \( M < \infty \), and let \( (x_j) \) be non-negative sequence in \( \ell_1(w) \). Then \( \sum_{j=1}^{\infty} u_j x_j \) is convergent, and we have

\[
\|Bx\|_{\ell_1(w)} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} b_{i,j} x_j
\]

\[
= \sum_{j=1}^{\infty} u_j x_j
\]

\[
\leq M \sum_{j=1}^{\infty} w_j x_j
\]

\[
= M \|x\|_{\ell_1(w)}.
\]

Hence \( \|B\|_{\ell_1(w)} = M \).

2. The Averaging Operator

Let \( A \) be the Averaging operator, given by \( Ax = y \), where

\[
y_n = \frac{1}{n} (x_1 + x_2 + \ldots + x_n).
\]

It is given by the Cesaro matrix:

\[
a_{n,k} = \begin{cases} \frac{1}{n} & \text{for } k \leq n \\ 0 & \text{for } k > n \end{cases}
\]

This is a lower triangular matrix. When \( A \) is regarded as an operator on \( \ell_p(p > 1) \), Hardy’s inequality [5] states that \( \|A\|_p = \frac{p}{p-1} \), and the lower bound \( L_p(A) \) is \( \zeta(p)^{1/p} \) [1]. (The element \( e_1 \) is enough to show that \( \bar{A} \) does not map \( \ell_1 \) into itself.

Note that \( \bar{A} e_1 \) is the sequence \( \left( \frac{1}{n} \right) \), so for it to belonging to \( \ell_1(w) \) we need \( \sum_{n=1}^{\infty} \frac{w_n}{n} \) to be convergent. Write

\[
u_n = \sum_{k=1}^{\infty} \frac{w_k}{k}
\]

and \( U_n = u_1 + \ldots + u_n \). It is easily verified that \( U_n = W_n + nu_{n+1} \).
Corollary: The operator $A$ maps $\ell_1(w)$ into $\ell_1(w)$ if

$$\sup_{n} \frac{U_n}{W_n} = \sup_n \left(1 + \frac{nn_{n+1}}{W_n}\right)$$

is finite and this supremum is then the norm of $A$.

In certain cases, it is enough to consider the sequence $\left(\frac{u_n}{w_n}\right)$ instead of $\left(\frac{U_n}{W_n}\right)$, because of the well-known facts listed in the following lemma.

Lemma 2: Suppose $u = (u_n)$ and $w = (w_n)$ are sequences of positive numbers.

(i) If $m \leq \frac{u_n}{w_n} \leq M$ for all $n$, then $m \leq \frac{U_n}{W_n} \leq M$ for all $n$.

(ii) If $\frac{u_n}{w_n}$ is increasing (or decreasing), then we have $\frac{U_n}{W_n} \rightarrow \infty$.

(iii) If $\frac{u_n}{w_n} \rightarrow U$ as $n \rightarrow \infty$, then $\frac{U_n}{W_n} \rightarrow U$ as $n \rightarrow \infty$ (also with $U = \infty$).

Proof: Elementary.

Proposition 5: Suppose $B$ is an operator satisfying (1), (2) and (3), and $\frac{u_n}{w_n} \rightarrow U$ as $n \rightarrow \infty$. Then $B$ maps $\ell_1(w)$ into $\ell_1(w)$.

Proof: By Lemma 2(i), since $\frac{u_n}{w_n} \leq U$ for each $n$, it follows that $\frac{U_n}{W_n} \leq U(nw)$. Also, applying Lemma 2(ii), we deduce that $U_n \rightarrow U \Longleftrightarrow n \rightarrow \infty$.

Since $\frac{u_n}{w_n} \leq U$, as $n \rightarrow \infty$, hence

$$M_{\ell_1(B)}(B) = \|B\|_{\ell_1(w)} = U.$$

Proposition 6: Let $B$ be an operator on $\ell_1(w)$ which satisfies conditions (1), (2) and (3).

If $u = (u_n)$ is such that $\frac{u_n}{w_n} \leq \frac{u_1}{w_1}$ for all $n$ (in particular, if $\left(\frac{u_n}{w_n}\right)$ is decreasing), then $B$ is an operator from $\ell_1(w)$ into $\ell_1(w)$, and also

$$\|B\|_{\ell_1(w)} = \frac{u_1}{w_1} = M_{\ell_1(B)}.$$

Proof: Since $\frac{u_n}{w_n} \leq \frac{u_1}{w_1}$ for each $n$, by Lemma 2(i), it follows that $\frac{U_n}{W_n} \leq \frac{u_1}{w_1}$.

Also, we have $\frac{U_1}{W_1} = \frac{u_1}{w_1}$, and hence

$$\|B\|_{\ell_1(w)} = \sup \frac{u_n}{w_n} = \frac{u_1}{w_1} = \sup \frac{U_n}{W_n} = M_{\ell_1(B)}.$$

Lemma 3: Suppose that $v_n > 0$, $u_n > 0$ for all $n$ and that $\sum_{n=1}^{\infty} v_n$ and $\sum_{n=1}^{\infty} u_n$ are convergent. Let $U_n = \sum_{j=1}^{\infty} u_j$, similarly $V_n$. If $\left(\frac{u_n}{v_n}\right)$ is increasing (or decreasing), then so is $\left(\frac{U_n}{V_n}\right)$.

Proof: Elementary.

Proposition 7: Let $r > 0$ and let $U_n = \sum_{j=1}^{\infty} \frac{1}{j^r}$. Then $n^nU_n$ decreasing, $(n-1)^rU_n$ increasing. Both tend to $\frac{1}{r}$ as $n \rightarrow \infty$.

Proof: Let $u_n = \frac{1}{n^r}$ and $v_n = \int_{n-1}^{n} \frac{1}{t^r} dt$.

Then $V_{n+1} = \frac{1}{r^n}$. By the usual integral comparison,

$$\frac{1}{r^n} \leq U_n \leq \frac{1}{r(n-1)^n},$$

which implies the stated limits. By Lemma 5, $\left(\frac{u_n}{v_n}\right)$ is decreasing, so by Lemma 2(ii), $\frac{U_n}{V_n} = \frac{1}{r(n-1)^r}$ is decreasing. Similarly, $\frac{U_n}{V_n}$ is increasing.

Remark: This is stated without proof in [4], Remark 4.10.

Theorem 2: If $w_n = \left(\frac{1}{p^r}\right)$, where $0 < p \leq 1$, then the weighted mean matrix $A_d$, where $d_n = \left(\frac{1}{d^n}\right)$ such that $0 < p - q \leq 1$, is a bounded operator from $\ell_1(w)$ into itself. Also, we have

$$M_{\ell_1(A_d)} = \|A_d\|_{\ell_1(w)} = \zeta(p-q+1),$$

where $\zeta(p-q+1) = \sum_{k=1}^{\infty} \frac{1}{k^p-q+1}$.

Proof: Since $u_n = \sum_{k=n}^{\infty} \frac{w_k d_n}{D_k}$, then we have

$$\frac{u_n}{w_n} = n^p \sum_{k=n}^{\infty} \frac{w_k d_n}{D_k} \leq n^{p-q} \sum_{k=n}^{\infty} \frac{1}{k^{p-q+1}} \leq \frac{u_1}{w_1}.$$

Now, by Proposition 6, it follows that

$$M_{\ell_1(A_d)} = \|A_d\|_{\ell_1(w)} = \frac{u_1}{w_1} = \zeta(p-q+1).$$
Corollary 2: ([7], Theorem 2.2.1) If \( w = (\frac{1}{n^p}) \), where \( 0 < p \leq 1 \), then the Cesaro operator \( A \) is a bounded operator from \( \ell_1(w) \) into \( \ell_1(w) \). Also, we have

\[
\|A\|_{\ell_1(w)} = \zeta(1 + p) = \|A\|_{w,1}.
\]

We now consider our second choice of weighting sequence, defined by \( W_n = n^{1-p} \) (where \( 0 < p < 1 \)), so that

\[
w_n = n^{1-p} - (n - 1)^{1-p} = \int_{n-1}^{n} \frac{1-p}{t^p} dt.
\]

Since \( W_n \) is now simpler than \( w_n \), we work with \( \frac{u}{W_n} \) instead of \( \frac{u}{w_n} \). In contrast to the previous case, we will show that this sequence is increasing.

We have

\[
U_n = w_1 + \ldots + w_n + n \sum_{k \geq n+1} \frac{w_k}{k}
\]

so

\[
\frac{U_n}{W_n} = 1 + \frac{n u_{n+1}}{W_n} = 1 + n^p u_{n+1}.
\]

Write \( v_n = \frac{1}{n^p (n+1)} \) and (as before) \( V(n) = \sum_{k \geq n} v_k \).

Lemma 4: With this notation, we have \( 1 + n^p u_{n+1} = n^p V(n) \) for \( n \geq 1 \). Also, we have \( u_1 = V(1) \).

Proof: By Abel summation,

\[
u_{n+1} = \sum_{k \geq n+1} \frac{w_k}{k} = \sum_{k \geq n+1} \left( \frac{1}{k} - \frac{1}{k+1} \right) W_k - W_n = \sum_{k \geq n} \left( \frac{1}{k} - \frac{1}{k+1} \right) W_k - W_n = \sum_{k \geq n} \frac{1}{k^p (k+1)} - \frac{1}{n^p}.
\]

The first statement follows. Moreover, \( u_1 = 1 + u_2 = V(1) \).

Theorem 3: Let \( v_n = \frac{1}{n^p (n+1)} \), where \( p > 0 \). Then \( n^p V(n) \to \frac{1}{p} \) as \( n \to \infty \). Also, \( n^p V(n) \) is increasing if \( 0 < p \leq 1 \) and decreasing if \( p > 1 \).

Hence if \( W_n = n^{1-p} \), then \( A \) is the Cesaro operator, then

\[
M_{w,1}(A) = \|A\|_{w,1} = \frac{1}{p}.
\]

Proof: Clearly,

\[
\frac{1}{(n+1)^{1+p}} \leq v_n \leq \frac{1}{n^{1+p}}.
\]

The stated limit follows, by Proposition 7. Also, by Lemma 3, we can prove monotonicity. Let \( s_n = \frac{1}{n^p} - \frac{1}{(n+1)^p} \), so that \( S(n) = \frac{1}{n^p} \).

\[
s_n = \frac{(n+1)^p - n^p}{n^p (n+1)^p} = \frac{(n+1)^p - n^p}{(n+1)^p - 1}\int_n^{n+1} t^{p-1} dt.
\]

This is decreasing if \( 0 < p < 1 \), increasing if \( p > 1 \). By Lemma 3, the same is true for \( \left( \frac{S(n)}{V(n)} \right) \).

We have now the last statement.

References