Exploring the concept of S-convexity

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Abstract: – The purpose of this paper is to distinguish, as much as possible, the concept of s-convexity from the concept of convexity and the concept of s-convexity in the first sense from the concept of s-convexity in the second sense. In this respect, the present work further develops a previous study by Orlicz(1961, [3]), Hudzik and Maligranda (1994, [1]).

Key-words [1]*:* convex, s-convex, function

1 Introduction

Recently, Hudzik and Maligranda ([1]) studied some classes of functions introduced by Orlicz ([3]), the classes of s-convex functions. Although they claim, in their abstract, to be providing several examples and to be clarifying the idea introduced by Orlicz further, their work leaves plenty of room to build over the concept.

The old conclusions presented here are:

1. theoretical definitions of convex/s-convex functions;
2. a theorem which acts as a generator of s-convex functions.

The new conclusions arisen from this paper are:

1. a rephrasing of the theoretical definitions of s-convex functions to look more similar to the definition of convex function;
2. some new symbols to represent the classes of s-convex functions;

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3. an identity between the class of 1-convex functions and the class of convex functions;

4. a conjecture about the the looks of a $s-$convex function;

5. some theorems on functions that are s-convex in both senses;

6. a few other side results that might suit future work or are, at least, useful to clarify similarities and differences between functions that are s-convex in the first sense and the ones which are s-convex in the second sense.

The paper is organized as follows: First, in section 2, we present the usual definition of both convex and s-convex functions. In section 3, we criticize the present presentation of definitions of s-convex functions. In section 4, we introduce a few new ways of referring to s-convex functions with views to have a more mathematical jargon to deal with them. In section 5, we re-write the definition of s-convex functions based on our new symbology, prove the equivalence between restrictions of convex functions and s-convex functions, and present some consequences of the definition of s-convex functions. In section 6, we recall one theorem on how to generate s-convex functions, as presented by Hudzik and Maligranda in [1]. Section 7 brings our conjecture whilst section 8 presents our conclusions.

[2]here, f means closure of $\Re$

2 The usual definition of convexity and s-convexity

The concept of convexity that is mostly cited in the bibliography is (as an example, [4]):

Definition 1. The function $(f : X- > \Re_f)^2$ is called convex if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

holds $\forall\lambda \in [0,1]$, $\forall x, y \in X$ such that the right-hand side is well defined. It is called strictly convex if the above inequality strictly holds $\forall\lambda \in]0,1[$ and for all pairs of distinct points $x, y \in X$ with $f(x) < \infty$ and $f(y) < \infty$.

In some sources, such as [2], convexity is defined only in geometrical terms as being the property of a function whose graph bears tangents only under it. In their words,

Citation 1. f is called convex if the graph lies below the chord between any two points, that is, for every compact interval $J \subset I$, with boundary ∂J, and every linear function L, we have

$$sup_J(f-L) = sup_{\partial J}(f-L)$$

One calls f concave if $-f$ is convex.

The concept of s-convexity, on the other hand, is split into two notions which are described below with the basic condition that $0 < s \leq 1$. ([1])

Definition 2. A function $f : [0,\infty)- > \Re$ is said to be s-convex in the first sense if $f(ax+by) \leq a^s f(x) + b^s f(y)$, $\forall x, y \in [0,\infty)$ and $\forall a, b \geq 0$ with $a^s + b^s = 1$.

Definition 3. A function $f : [0, \infty) - > \Re$ is said to be s-convex in the second sense if $f(ax + by) \leq a^s f(x) + b^s f(y)$, $\forall x, y \in [0, \infty)$ and $\forall a, b \geq 0$ with $a + b = 1$.

3 What are the criticisms to the present definition of s-convexity?

- It seems that there is lack of objectivity in the present definition of s-convexity for there are some redundant things;
- It takes us a long time, the way the definition is written now, to work out the true difference between convex and s-convex functions;
- So far, we did not find references, in the bibliography, to the geometry of an s-convex function, what, once more, makes it less clear to understand the difference between an s-convex and a convex function whilst there are clear references to the geometry of the convex functions.

4 New Symbology

- In this paper, we mean that f is an s-convex function in the first sense by saying that $f \in K_s^1$;
- We use the same reasoning for a function g, s-convex in the second sense and say then that $g \in K_s^2$;
- We name s_1 the generic class constant for those functions that are s-convex in the first sense;
- We name s_2 the generic class constant for those functions that are s-convex in the second sense.

5 The first few new results

5.1 Re-writting the definition of s-convex function

It is trivial to prove that $a, b \in [0, 1]$ is a consequence of the present definition of s-convexity.

Lemma 5.1. *If $f \in K_s^1$ or $f \in K_s^2$ then*

$$f(au + bv) \leq a^s f(u) + b^s f(v)$$

with $a, b \in [0, 1]$, exclusively.

Proof. We present the proof for K_s^1 only, since the proof for K_s^2 is analogous.
For K_s^1: We first prove that it is not the case that $a > 1$ and $b > 1$. Supposing that it is the case that $a > 1$ and $b > 1$, that implies having

$$a = 1 + \epsilon$$

$$b = 1 + \delta$$

$$a^s + b^s = 1, 0 < s \leq 1$$

Therefore,

$$(1 + \epsilon)^{\frac{1}{n}} + (1 + \delta)^{\frac{1}{n}} = 1, 1 \leq n < +\infty$$

As $x^{\frac{1}{n}}$ is a decreasing function of n, for $x > 1$, and, as $n- > +\infty$, the above result is not verified, being $a^s + b^s > 1$, $\neg(a > 1 \wedge b > 1)$. Secondly, we prove that it is not the case that $a > 1$ and $b < 1$, or vice-versa, just by re-analyzing the previous case again. Therefore, $\neg(a < 1 \wedge b > 1) \wedge \neg(a > 1 \wedge b < 1)$. Thirdly, we conclude that it must be the case that $(a \leq 1 \wedge b \leq 1)$. But since the definition of s-convexity uses $a, b \geq 0$, we have that

$$a, b \subset [0, 1]$$

□

With this, we may re-write the definitions of s-convexity in each of the senses as being:

Definition 4. A function $f : X- > \Re$ is said to be s-convex in the first sense if $f(\lambda x + (1 - \lambda^s)^{\frac{1}{s}} y) \leq \lambda^s f(x) + (1 - \lambda^s) f(y)$, $\forall x, y \in X$ and $\forall \lambda \in [0, 1]$ where $X \subset \Re_+$.

Definition 5. A function $f : X- > \Re$ is said to be s-convex in the second sense if $f(\lambda x + (1 - \lambda) y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y)$, $\forall x, y \in X$ and $\forall \lambda \in [0, 1]$ where $X \subset \Re_+$.

6 The classes K_1^1, K_1^2, and convex coincide when the domains are restricted to $\Re_+$

Theorem 6.1. *The classes K_1^1, K_1^2, and convex are equivalent when the domain is restricted to $\Re_+$.*

Proof. Just a matter of applying the definitions. □

Natural implication: All $1-$convex functions are convex.

7 Some natural consequences of the definition of s-convex functions

Theorem 7.1.

$$f \in K_s^1 \implies f\left(\frac{u+v}{2^{\frac{1}{s}}}\right) \leq \frac{f(u)+f(v)}{2}$$

Proof. Simply consider the case where $a^s = b^s = \frac{1}{2}$. □

Theorem 7.2.

$$f \in K_s^2 \implies f\left(\frac{u+v}{2}\right) \leq \frac{f(u)+f(v)}{2^s}$$

Proof. Simply consider the case where $a = b = \frac{1}{2}$. □

Theorem 7.3. *For a function that is both s_1 and s_2-convex, there is a perfect bijection between the set of (a's,b's) used in s_1 and the set of (a's, b's) used in s_2.*

Proof. Each a may be written as an a_1^s and each b as a b_1^s and vice-versa. This happens because $a, b \in [0, 1]$, $s \in [0, 1]$(each $\frac{1}{s}$-root in $(0, 1)$ will give us a number in $(0, 1)$). □

Theorem 7.4. *If a function belongs to both K_s^1 and K_s^2, then*

$$f(a_1 u + b_1 v) \leq a_1^s f(u) + b_1^s f(v) \leq a_2^s f(u) + b_2^s f(v)$$

for some $\{a_1, b_1, a_2, b_2\} \subset [0, 1]$ and such that it occurs to each and all of them.

Proof. It follows from the bijection proved before. For each a_2, b_2 such that $a_2 + b_2 = 1$, it corresponds a_1, b_1 such that $a_1^s + b_1^s = 1$ and $a_2 \geq a_1$, $b_2 \geq b_1$ since $\{a, b\} \subset [0, 1]$. □

Theorem 7.5. *If a function belongs to both* K_s^1 *and* K_s^2 *and its domain coincides with its counter-domain then the composition* $f(f)$ *is* s_1^2*-convex.*

Proof. $f(a_1 u + (1 - a_1^s)^{\frac{1}{s}} v) \leq a_1^s f(u) + (1 - a_1^s) f(v) \implies f(a_1^s f(u) + (1 - a_1^s) f(v)) \leq (a_1^s)^s f(f(u)) + (1 - a_1^s)^s f(f(v)) = a_2^s f(f(u)) + b_2^s f(f(v))$ □

Theorem 7.6. $f : I- > \Re$*,* $I \subset [0, \infty)$*,* f *being a convex, non-negative function, then* $\forall s \in (0, 1]$*,* f *is* s_2*-convex.*

Proof.

$$a + b = 1$$

$f(ax + by) \leq af(x) + bf(y) \leq a^s f(x) + b^s f(y)$ □

8 A new conjecture

Taking into account the relationship between a^s and a, we may wonder whether the following is true or not:

Conjecture 1. f is called $s-$convex if the graph lies below the 'bent chord' between any two points, that is, for every compact interval $J \subset I$, with boundary ∂J, and every linear function L, we have

$$G(s) \geq sup_J(f - L) \geq sup_{\partial J}(f - L)$$

9 Conclusions

In this paper, we proved that s-convexity may be stated in a very similar way to convexity, as written below:

Definition 6. the function $(f : X- > \Re_f)$ [3] is called convex if the inequality

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

holds $\forall \lambda \in [0, 1]$, $\forall x, y \in X$.

For $0 < s_1, s_2 \leq 1$,

Definition 7. A function $f : X- > \Re$ is said to be s_1-convex if the inequality

$$f(\lambda x + (1 - \lambda^s)^{\frac{1}{s}} y) \leq \lambda^s f(x) + (1 - \lambda^s) f(y)$$

holds $\forall \lambda \in [0, 1]$, $\forall x, y \in X$ such that $X \subset \Re_+$.

Definition 8. A function $f : X- > \Re$ is said to be s_2-convex if the inequality

$$f(\lambda x + (1 - \lambda) y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y)$$

holds $\forall \lambda \in [0, 1]$, $\forall x, y \in X$ such that $X \subset \Re_+$.

The own re-definition of s-convexity included our new way of referring to s-convex functions by creating class-like symbology for them:

- K_s^1 for the class of s-convex functions in the first sense, some s;
- K_s^2 for the class of s-convex functions in the second sense, some s;
- K_0 for the class of convex functions;
- s_1 for the constant s, $0 < s \leq 1$, used in the first definition of s-convexity;

[3] here, f means closure of $\Re$

- s_2 for the constant s, $0 < s \leq 1$, used in the second definition of s-convexity. thirdly, we pointed out that the class of 1-convex functions is just a restriction of the class of convex functions, that is, when $X = \Re_+$,

$$K_1^1 \equiv K_1^2 \equiv K_0$$

In fourth, we introduced the following side-theorems:

Theorem 9.1. *For a function that is both s_1 and s_2-convex, there is a perfect bijection between the set of (a's,b's) used in s_1 and the set of (a's, b's) used in s_2.*

Theorem 9.2. *If a function belongs to both K_s^1 and K_s^2, then*

$$f(a_1u+b_1v) \leq a_1^s f(u)+b_1^s f(v) \leq a_2^s f(u)+b_2^s f(v)$$

for some $\{a_1, b_1, a_2, b_2\} \subset [0,1]$ obeying K_s^1 and K_s^2 rules, and such that it occurs to each and all of them.

Theorem 9.3. *If a function belongs to both K_s^1 and K_s^2 and its domain coincides with its counter-domain then the composition $f(f)$ is s_1^2-convex.*

Theorem 9.4. *$f : I- > \Re$, $I \subset [0, \infty)$, f being a convex, non-negative function, then $\forall s \in (0, 1]$, f is s_2-convex.*

In fifth we bring our conjecture as a prospective future work:

Conjecture 2. f is called $s-$convex if the graph lies below the 'bent chord' between any two points, that is, for every compact interval $J \subset I$, with boundary ∂J, and every linear function L, we have

$$G(s) \geq sup_J(f - L) \geq sup_{\partial J}(f - L)$$

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