A New Stochastic Framework for Modelling Optical Communication Systems

PÉTER VÁRLAKI, LÁSZLÓ NÁDAI
Budapest University of Technology and Economics
H-1111 Budapest, Bertalan L. u. 2.
HUNGARY

MIKLÓS BULLA
Széchenyi István University
H-9026 Győr, Egyetem tér 1.
HUNGARY

ANIKÓ SZAKÁL
Budapest Polytechnic
H-1081 Budapest, Népszínház u. 8.
HUNGARY

Abstract: A stochastic control process model is presented for simulating the output signal of an optical interferometer connected to light source of known (output) statistics. A probabilistic method is given by statistical analysis of the time dependent vector signal process, using the covariance characteristics of the interferometer (together with the "photon space") that can be described by the models of control theory. The developed and applied mathematical methods are highly determinate by the mechanism of the detector device.

Keywords: Optical communication, stochastic modelling, statistical identification.

1 Introduction

The question "whether a photon is a particle or a wave" has been raised for decades. Since both qualities can be shown with a proper experimental arrangement, the question had only historical interest, because quantum optics is capable of describing both properties of the photon.

For a long time it seemed that such questions would have been forgotten, but in the last two decades the development of laser metrology (and certain measurements in particle physics, see [1,2,4]) made it possible to detect low intensity light and particle beams. Problems like EPR, "which way", "quantum eraser" etc. were raised again, because it could not longer been stated that "the measurement is applied to a statistical sample". For up to this point only correlation, coincidence and momentums had been measured, the model of signal process based on the results of stochastic control theory was not involved.

Our attention was turned to observing and modelling an optical process which is capable of producing (measurable) interference, because this is probably the simplest complex problem in modern physics where the aforementioned aspects are present simultaneously. Furthermore, not only the arrangement of experiment is easy technically, but the measuring device can also be clearly described theoretically (even classically).

We would like to base our investigation theoretically on stochastic dependencies between absorption processes in the elements of the detector array. As the interference picture, that is the average intensity distribution, can not be interpreted after the impact of some photons, this paradox can be lift by applying double stochastic vector processes, which are well-known from modern probability theory. Moreover, we can give a statistical characterization for the dynamics of absorption time (events of photon detection) vector processes

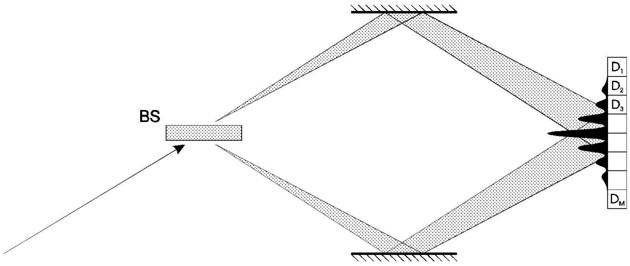

Figure 1: Mach–Zehnder interferometer (BS: Beam Splitter, M: mirror, $D_1, \ldots, D_M$: Detector Array)

by assuming the existence of stochastic connection with "hidden" generating information processes.

2 The physical model

Let us consider a Mach–Zehnder interferometer, see Figure 1. The detector array constructed from semiconductor pixels, in which electrons are excited to the conductance band by the absorbed photons.

The output signal (charge) of pixels can be read out by $\Delta\tau$ time intervals, according to the clock signal of the measurement card. (Obviously, we have to take into account the wake-up time and dark noise of detector pixels in determining $\Delta\tau$.)

Let us denote the output signal of the ith pixel by $X_n^{(i)}$ ($i = 1, \ldots, M$, $n = 1, 2, \ldots$) which – choosing an appropriate average of the signals of detectors be the unit – gives integer numbers during the reading time, thus the signal of the detector array (neglecting the reading and detector noises) at nth reading is

$$X_n = \left(X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(M)}\right), \qquad n = 1, 2 \ldots, \tag{1}$$

where M is the number of pixels. Denote the arrival time of the jth photon on ith pixel by $\tau_j^{(i)}$, $j = 1, 2, \ldots$. Now define $Z(t)$ counter vector process as

$$Z(t) = \left(Z^{(1)}(t), Z^{(2)}(t), \ldots, Z^{(M)}(t)\right), \tag{2}$$

where

$$Z^{(i)}(t) = \sum_{j=1}^{\infty} I\left(\tau_j^{(i)} \le t\right).$$

(I is the indicator function.) $\{\tau_j^{(i)}, j = 1, 2, \ldots\}$ absorption time series can also be expressed using the measurement series $\{X_n^{(i)}, n = 1, 2, \ldots\}$, $1 \le i \le N$:

$$X_n^{(i)} = \sum_{j=1}^{\infty} I\left((n-1)\Delta\tau < \tau_j^{(i)} \le n\Delta\tau\right) = = Z^{(i)}(n\Delta\tau) - Z^{(i)}((n-1)\Delta\tau). \tag{3}$$

Now $Z^{(i)}(t)$ can be modelled by a doubly stochastic Poisson process.

Definition 1 *For a single detector*

$$Z(t) := \{N_t; t \ge t_0\} \tag{4}$$

is a doubly stochastic Poisson process [3] with intensity process $\{\lambda_t(\mathbf{y}_t); t \ge t_0\}$ *if for almost every given path of the process* $\{\mathbf{y}_t; t \ge t_0\}$*; N is a Poisson process with intensity function* $\lambda_t(\mathbf{y}_t)$*. In other words,* $\{N_t; t \ge t_0\}$ *is conditionally a Poisson process with intensity function* $\lambda_t(\mathbf{y}_t)$ *given* $\{\mathbf{y}_t; t \ge t_0\}$.

Remark 1 *The process* $\{\mathbf{y}_t; t \ge t_0\}$ *we shall encounter as it conveys desired information, and for this reason we call it the information process, which in our case will be called as propensity (output) process.*

Our model for a so-called multichannel (multi-detector) doubly stochastic Poisson process is comprised of M single-channel doubly stochastic Poisson processes that are conditionally mutually independent Poisson processes given the information process. Thus, let $\{N_t^{(m)}; t \ge t_0\}$ for $m = 1, 2, \ldots, M$ be doubly stochastic Poisson process with corresponding intensity processes $\{\lambda_t^{(m)}(\mathbf{y}_t); t \ge t_0\}$ for $m = 1, 2, \ldots, M$, where $\{\mathbf{y}_t; t \ge t_0\}$ is an information process. We assume $N^{(1)}, N^{(2)}, \ldots, N^{(M)}$ are mutually independent given $\{\mathbf{y}_t; t \ge t_0\}$. We term the vector $\mathbf{N}_t$, where

$$\mathbf{N}_t := \left[N_t^{(1)}, N_t^{(2)}, \ldots, N_t^{(M)}\right]', \tag{5}$$

a multichannel doubly stochastic Poisson process with intensity process $\lambda_t(\mathbf{y}_t)$, where

$$\lambda_t(\mathbf{y}_t) := \left[\lambda_t^{(1)}(\mathbf{y}_t), \lambda_t^{(2)}(\mathbf{y}_t), \ldots, \lambda_t^{(M)}(\mathbf{y}_t)\right]'. \tag{6}$$

Let us consider the information (propensity) process being an M dimensional complex valued Gauss stationary series $\{\mathbf{y}_t = (y_t^{(1)}, \ldots, y_t^{(M)}; t \ge t_0\}$, and using this the intensity process can be characterized. The identification of the Gauss type information process $\{\mathbf{y}_t; t \ge 0\}$ from absorption times needs the application of further complex mathematical tools. We neglect this, instead we consider the analysis of the stochastic dependencies between absorption and information processes, which forms the base for the statistical identification of the information process.

It is of special interest, that in case of highly weakened light beams the intensity and information processes can not be identified directly in the base time

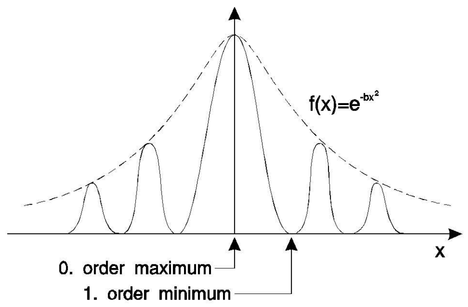

Figure 2: Qualitative illustration of interference picture

slot, as opposed to normal intensity beams, when having the appropriate a priori physical (technical) information intensity and information processes can be estimated quite easily. This gives us the reason for considering the information process as propensity process. Further, it concludes that for the characterization of dependence between arrival processes on individual detectors the usual measures of dependence can not be used, instead special techniques are adequate (see [5]).

Let us consider the process for the interference picture illustrated in Figure 2. From symmetry it is reasonable that in this case the individual time scales t_i are uniquely defined by the geometric arrangement of the detector array and by the number of interference stripes to be processed.

The general formula is $e^{-bx^2}\cos bx$, if the coherence length is great enough compared to the wave length, i.e. if $l_{\text{coh}} \gg \lambda$ ($l_{\text{coh}} \approx 3-5$ cm) then the intensity distribution of the interference picture is

$$I(x) \approx I_0 + I' \cos^2(kx), \tag{7}$$

where I_0 constant, and I' can be considered as constant ($\Delta I'$ is the error of intensity measurement, $\Delta I' < I'/100$ can be achieved easily by tuning the measuring arrangement; greater exactness is an unrealistic expectation because of the inhomogeneities in optical elements).

On the basis of Figure 3. we can write that

$$P_{i,\Delta x} \sim \int_{x_i}^{x_i+\Delta x} I_0 + I' \cos^2(kx)dx, \tag{8}$$

therefore

$$P_{i,\Delta x} \sim \frac{I_0 \Delta x}{k} + \frac{I'\Delta x}{k} + \frac{I'}{k}\big[\cos(2k(x_i+\Delta x)) - \cos(2kx_i)\big], \tag{9}$$

where Δx is the width or height of a detector pixel.

Remark 2 *Because of difficulties in controlling the transversal behavior of the measuring arrangement*

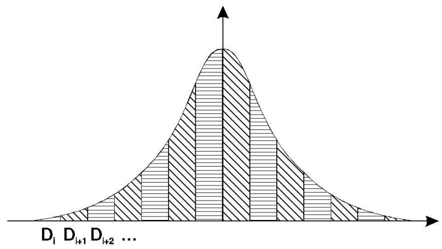

Figure 3: Intensity distribution

there can arise arbitrary phases, but they do not modify the result significantly:

$$P_{i,\Delta x} \sim \frac{I_0 \Delta x}{k} + \frac{I'\Delta x}{k} + \frac{I'}{k}\big[\cos(2k(x_i+\Delta x)+\phi) - \cos(2kx_i+\phi)\big]. \tag{10}$$

Probabilities defined in interval $[0,1]$ can be derived by the appropriate normalization of the above equations, e.g.

$$p_i = p_{i,\Delta x} = \frac{P_{i,\Delta x}}{\sum_j P_{j,\Delta x}}. \tag{11}$$

3 The statistical model

Let us examine the interpretation of intensity processes on the different possible time scales. Let $n_k = 1/p_k$-t ($k = 1,\dots,M$) be the interval length when the average number of photon absorptions in pixel k is 1. In the following we fix this scale.

In our case it is obvious that individual occurrence times can not be observed. Then let $\{N_t^{(i)}, t \ge t_0\}$ be a doubly stochastic Poisson process with intensity process $\{\lambda_t^{(i)}, t \ge t_0\}$. Assume that the mean $\mathbb{E}(\lambda_t^{(i)})$ and covariance function $K_{\lambda^{(i)}\lambda^{(j)}}(t,u)$ for the double stochastic Poisson intensity process are known. An observation interval $[t_0,T]$ is partitioned into $m^{(i)} = [T - t_0/n_i]$ subintervals according to the times $t_0^{(i)} < t_1^{(i)} < t_2^{(i)} < \cdots < t_m^{(i)} = T$, $t_k^{(i)} = kn_i\Delta\tau$, $k = 0,1,\dots,m^{(i)}$; and the number of points occurring in each subinterval is observed. Denote these observables by $W_1^{(i)}, W_2^{(i)}, \dots, W_k^{(i)}$, $i = 1,2,\dots,M$, where

$$W_k^{(i)} = N^{(i)}\left(\lambda_{kn_i\Delta\tau}^{(i)}\right) - N^{(i)}\left(\lambda_{(k-1)n_i\Delta\tau}^{(i)}\right) = = \sum_{l=(k-1)n_i+1}^{kn_i} x_j^{(i)}, \qquad k = 1,2,\dots,m^{(i)}, \tag{12}$$

here $x_j^{(i)}$ was defined by (3) for the measurement data "interpretation" of the physical model. $\nu_k^{(i)}$ is defined theoretically as

$$\nu_k^{(i)} = \int_{(k-1)n_i\Delta\tau}^{kn_i\Delta\tau} \lambda_s^{(i)} ds, \quad \text{and}$$
$$\hat{\nu}_k^{(i)} = \left(t_k^{(i)} - t_{k-1}^{(i)}\right)^{-1} \hat{\nu}_k^{(i)}. \tag{13}$$

Remark 3 *Since filtering N_t for determination of λ_t is not possible (because we do not know a priori and can not estimate directly covariance function K), we must estimate discrete intensity process ν from the observed series of $W_{\tau k}^i$.*

Thus, the covariance function of $\nu_j^{(i)}$ is

$$R_{ij}(u_k, s_l) = \operatorname{cov}\left(\nu_j^{(i)}, \nu_h^{(i)}\right) = \\ = \int_{t_{k-1}}^{t_k} \int_{t_{l-1}}^{t_l} K_{ij}(\tau, \sigma) d\tau d\sigma, \tag{14}$$

and

$$\hat{R}_{ij}(u_k, s_l) = (t_k - t_{k-1})^{-1}(t_l - t_{l-1})^{-1} R_{ij}(u_k, s_l). \tag{15}$$

In this case obviously

$$\mathbb{E}\left(W_k^{(i)}\right) = \mathbb{E}\left(\nu_k^{(i)}\right) = (t_k - t_{k-1})\mathbb{E}\left(\hat{\nu}_j^{(i)}\right), \tag{16}$$

where $\mathbb{E}$ is the symbol of mathematical expectation, and

$$\operatorname{cov}\left(W_k^{(i)}, W_l^{(i)}\right) = \tag{17}$$
$$= \begin{cases} \operatorname{cov}\left(\nu_k^{(i)}), \nu_l^{(i)}\right), & \text{for } k \neq l, \\ \operatorname{cov}\left(\nu_k^{(i)}, \nu_l^{(i)}\right) + \mathbb{E}\left(\nu_j^{(i)}\right), & \text{for } k = l,\, i = j. \end{cases}$$

Further, examine the coupled behavior of two detectors (i and j, $1 \leq i, j \leq M$) for some fixed scale $n_i = n_j = n$ (here the selection of $n \sim 1/p_i$ and $n \sim 1/p_j$ plays important role), but we omit the explicit notation on n for the sake of simplicity. Then the estimation of discrete intensity process $\nu_k^{(i)}$ can be determined by the following filtering formula

$$\nu_k^{(i)} = \mathbb{E}\left(\nu_k^{(i)}\right) + \sum_{j=1}^{M} \sum_{l=1}^{m} g_l^{(j)} W_{k-l}^{(j)}. \tag{18}$$

Here the optimal estimation (e.g. in minimal mean square) of the discrete intensity process $\nu_k^{(i)}$ can be carried out by the well-known methods (e.g. correlation equation system, state-space model). On the basis of the estimated discrete intensity process $y_k^{(i)}$ complex-valued Gauss stationary time series can be defined by expression

$$\nu_k^{(i)} = \mathbb{E}\left(\nu_k^{(i)}\right) + \left(\Re y_k^{(i)}\right)^2 - \left(\Im y_k^{(i)}\right)^2. \tag{19}$$

After the concrete computation of $y_k^{(i)}$ from $\nu_k^{(i)}$ the system identification of $y_k^{(i)}$ generated by forward and backward (or acausal) vector white noise processes can be carried out (for the different time scales) by rather sophisticated structure and parameter estimation methods, that is a subject of a forthcoming paper.

Acknowledgements

This research work was partially supported by the Hungarian Scientific Research Fund under grant no. OTKA T 043177.

References

[1] Aspect, A., J. Dalibard and G. Roger: *Phys. Rev. Lett.*, 22, 1982, 1804.

[2] Pittman, T.B., D.V. Strekalov, A. Migdall, M.H. Rubin, A.V. Sergienko and Y.H. Shih, *Phys. Rev. Lett.*, 77(10), 1996, 1917–1920.

[3] Snyder, D.L., *Random point processes*. Wiley & Sons, New York, 1975.

[4] Ou, Z.Y. and L. Mandel, *Phys. Rev. Lett.*, 61, 1988, 50.

[5] Várlaki, P., L. Szeidl and L. Nádai, "New stochastic approach for measuring and controlling photon statistics". *Proc. of Workshop AUTOMATION 2001*, Vienna, 1997, pp. 253–256.