On Solvability for Higher Order Parabolic Equations

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Abstract: - We consider the Cauchy problem for higher order linear parabolic equations in a layer. We establish new a priori estimates for the solution to this problem in general anisotropic Hölder norm, under the assumption that the coefficients and the independent term satisfy the general anisotropic Hölder condition of exponent $\alpha(l)$ with respect to the space variables only. In this connection, however, we also obtain an estimate for the modulus of continuity with respect to the time of the higher derivatives with respect to space variables of the corresponding solutions. On the basis of our new a priori estimates for the solution to the Cauchy problem for higher order linear parabolic equations, we establish the corresponding solvability theorem for this problem in general Hölder anisotropic spaces. We establish the local solvability of the Cauchy problem for higher order nonlinear parabolic equations with the aid of the results of the corresponding linear theory.

Key-Words: - estimates, solvability, equations, parabolicity, solution, problem

1. Introduction

In the present work we consider the higher order linear parabolic equation

$$\mathbf{u_t - \sum_{|k|\le 2m} a_k(t,x) D_x^k u = f(t,x)} \qquad (1)$$

in the layer $\Pi_T = [0,T] \times E_n$ with the initial condition

$$u|_{t=0} = \varphi(x) \qquad (2)$$

Here $x = (x_1,...,x_n)$ is a point of the n-dimensional Euclidean space E_n, $t \in [0,T]$.

$$\mathbf{k = (k_1,...,k_n),\ |k| = k_1 + ... + k_n,}$$

$$\mathbf{k_i \ge 0,\quad i = 1,...,n,\quad u_t = \frac{\partial u}{\partial t},}$$

$$\mathbf{D_x^k u = \frac{\partial^{|k|} u}{\partial x_1^{k_1} ... \partial x_n^{k_n}}}$$

We establish new a priori estimates for solutions $u(t,x)$ to the problem (1),(2) in general anisotropic norms, under the assumption that the coefficients) and the independent term are continuous satisfy the general Hölder in Π_T of exponent $\alpha(l), l > 0$ with respect to the space variables $x = (x_1,...,x_n)$ only (See theorem 1). In this connection, however, we also obtain an estimate for the modulus of continuity with respect to the time **t** of the leading derivatives $D_x^k u,\ \ |k| = 2m$ (See theorems 2,3).

Note that in the works [1]-[12] , the a priori estimates of this type have been obtained under the fulfilment of a (general) Hölder condition with respect to the totality of variables **(t,x)** on the coefficients and the independent term of equation (1).

On the basis of our new a priori estimates for the solutions to the problem (1), (2), we establish the corresponding solvability theorems for this problem in general Hölder anisotropic spaces (See theorem 4).

We assume that the coefficients of Equation (1) satisfy the uniform parabolicity condition: for any non-zero vector $\xi = (\xi_1,...,\xi_n) \in E_n\ ; (t,x) \in \Pi_T$

$$(-1)^{m+1}\sum_{|k|=2m} a_k(t,x)\xi^k > \lambda|\xi|^{2m},$$

$$\lambda = const. > 0,\ \xi^k = \xi_1^{k_1}\dots\xi_n^{k_n} \qquad (3)$$

We apply our results in the linear theory to establish the local solvability with respect to the time t, in general Hölder anisotropic spaces, to the Cauchy problem for the nonlinear parabolic equation (See theorem 5),

$$u_t = A(t,x,u,D_x u,\dots,D_x^{2m}u) \qquad (4)$$

in Π_T with the initial condition (2), where $D_x u = (\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n})$ and $D_x^r u$ is the set of all derivative $D_x^k u,\quad |k| = r,\ 1 \le r \le 2m$. In the present work, the equation (4) is linearized directly. No conditions are imposed here on the nature of the growth of the nonlinearity of the function $A(t,x,p^0,p^1,\dots,p^{2m})$ (See [9]),where p^0 scalar, $p^r = (\dots,p_k^r,\dots)$,which is defined for $(t,x)\in \Pi_T$ and any $p^0, p^r, 1 \le r \le 2m$.

The main assumption concerning to the function $A(t,x,p^0,p^1,\dots,p^{2m})$is the parabolicity condition: for any non zero vector $\xi = (\xi_1,\dots,\xi_n)\in E_n\ ;\ (t,x)\in \Pi_T$ and $p^0,p^1,\dots,p^{2m}$

$$(-1)^{m+1}\sum_{|k|=2m} A_{p_k^{2m}}(t,x,p^0,p^1,\dots,p^{2m})\xi^k > 0,$$

$$\xi^k = \xi_1^{k_1}\dots\xi_n^{k_n} \qquad (5)$$

In almost all the work we suppose that in the equation (1), the function $f = f_1 + f_2$;the functions $a_k(t,x), |k| = 2m$ and f_1 satisfy the general Hólder condition in Π_T of exponent $\beta(l), l > 0$ with respect to the space variables $x = (x_1,\dots,x_n)$ only and f_2 satisfies the general Hölder condition in Π_T of exponent $\alpha(l), l > 0$ with respect to the space variables $x = (x_1,\dots,x_n)$ only.

All the coefficients and the independent terms of equation 1 are continuous in the layer Π_T.

We require less smoothness conditions from the functions $A(t,x,p^0,p^1,\dots,p^{2m})$ and $\varphi(x)$ that in the works b[6] , [7] and [9] (See theorem 5).Some close results for second order parabolic equations have been established in [13] -[17] and [19].

2. Basic Notations. Auxiliary Propositions

We shall say that the function $u(t,x)$ defined in the layer Π_T satisfies the general Hölder condition of exponent $\beta(l), l > 0$ in Π_T with respect to the space variables if there exists a constant $C > 0$ such that

$$|u(t,x) - u(t,y)| \le C|x-y|^{\beta(|x-y|)},$$

$$(t,x),(t,y)\in \Pi_T.$$

The function $\beta(l)$ is defined and continuous in $0 < l < \infty$. Moreover it has the following properties:

Ia.

$$\beta(l) \to \sigma \in [0,1[$$

$$\text{if } l \to 0^+ \quad \text{or} \quad l \to +\infty$$

Ib.

$$\beta'(l)\, l \ln \to 0$$

$$\text{if } l \to 0^+ \quad \text{or} \quad l \to +\infty$$

Ic. If $\sigma = 0$ then

$\beta(l)\ln \to -\infty \quad for\ l \to 0^+$ and

$$(\beta(l) + \beta'(l)\, l \ln) > 0 \quad for \quad l \in]0.\, l_0[$$

where l_0 is sufficiently small number (we suppose that the derivative $\beta'(l)$ exists and it is a continuous function in $R_0 =]0.\, l_0[\cup]1/l_0, +\infty[$).

Note that the condition Ic introduces a new set of functions (the functions $u(t,x)$ that satisfy the general Hölder condition with respect to space variables only).In this new set of functions we will obtain the corresponding existence and uniqueness theorems for the solutions to the problems (1),(2) and (4),(2).

It follows from the conditions Ia, Ib, Ic that

IIa. $l^{\beta(l)}$ is a monotonically increasing function for $l \in R_0$.

IIb. $\frac{(ql)^{\beta(l)-\sigma}}{l^{\beta(l)-\sigma}} \to 1$ if and only if $l \to 0^+$ or $l \to +\infty$, uniformly respect to $q,\ 0 < a \le q \le b < \infty$

We denote by Γ the set of functions $\beta(l),\ l > 0$ for which

$$\Lambda\, l^{\beta(l)} \equiv \Lambda_{\beta(l)} \equiv \int_0^t t^{\beta(t)-1} dt < \infty$$

For the functions $\beta(l) \in \Gamma$ we introduce the functions $B_{\beta(l)} = \frac{1}{\ln l} \ln \frac{\Lambda_{\beta(l)}}{\Lambda_{\beta(1)}}$.

The function $B_{\beta(l)}$ is a function of the type $\beta(l)$.

IIIa. $B_{\beta(l)} > 0$ for $l \in]0.+\infty[$.

IIIb. There exist constants C_1, C_2 such that for $l \in R_0$

$$l^{\beta(l)} < C_1 l^{B_{\beta(l)}}$$

and

$$l^{B_{\beta(l)}} < C_2 l^{\beta(l)} \quad for \quad l > 0$$

IIIc. There exists a constant C_3 such that for $l \in]0.\, l_0[$

$$l \int_l^{+\infty} \rho^{\beta(\rho)-2} d\rho \le C_3 \Lambda_{\beta(\rho)}$$

Now we give two examples of functions of the type $\beta(l)$:

1 $\beta(l) = \beta,\ \beta = const.,\quad 0 < \beta < 1$

2.

$$\beta(l) = \begin{cases} \beta - \frac{b \ln(-\ln l)}{\ln l}, & 0 < l \le exp(-\frac{b}{a}) \\ \beta - \frac{b \ln(\frac{b}{a})}{\ln l}, & l > exp(-\frac{b}{a}) \end{cases}$$

where $0 < a < 1,\quad \beta = const.,\quad 0 \le \beta < 1$ (see [18]).

For the functions $u(t,x)$ defined and Hölder continuous(in the general sense) of exponent $\beta(l),\ l > 0$ in the layer Π_T with respect to space variables, we introduce the following norms

$$|u|_{0,0}^{[t]} = \sup_{0\le\tau\le t} |u|_{0,0}^{\tau},$$

$$|u|_{0,\beta(l)}^{[t]} = \sup_{0\le\tau\le t} |u|_{0,\beta(l)}^{\tau}, \tag{6}$$

where

$$|u|_{0,0}^{t} = \sup_{x\in E_n} |u(t,x)|,$$

$$|u|_{0,\beta(l)}^{t} = |u|_{0,0}^{t} + H_{\beta(l)}^{t}(u), \tag{7}$$

$$H_{\beta(l)}^{t}(u) = \sup_{\substack{x,y\in E_n \\ x\ne y}} \frac{|u(t,x)) - u(t,y)|}{|x-y|^{\beta(|x-y|)}} \tag{8}$$

For the functions $u(t,x)$ that have continuous derivatives with respect to x up to the order $m\ (m = 0,1,2,...)$ inclusively in the layer Π_T and satisfying the general Hölder condition of exponent $\beta(l),\ l > 0$, with respect to space variables in the layer Π_T we define the norms

$$|u|_{m,\beta(l)}^{\tau} = \sum_{|k|\le m} |D_x^k u|_{0,\beta(l)}^{\tau} \tag{9}$$

$m = 0,1,2,....$

$$|u|_{m,\beta(l)}^{[t]} = \sup_{0\le\tau\le t} |u|_{m,\beta(l)}^{\tau},\ m = 0,1,2,.... \tag{10}$$

We will denote by $C_{m,\beta(l)}^{[t]}(\Pi_t)\quad m = 0,1,2,...$ the Banach space of functions $u(t,x)$ that are continuous in $\Pi_t = [0,t] \times E_n$ together with all derivatives respect to x up to the order $m\,(m = 0,1,2,...)$ inclusively and have a finite norm (10).

With respect to the coefficients of the equation (1) we assume that

$$a_k(t,x) \in C_{0,\beta(l)}^{[t]}(\Pi_T)\quad,$$

$$0 \le |k| \le 2m$$

and

$$\sum_{|k|\le m} \left| a_k(t,x) \right|_{0,\beta(l)}^{[T]} = B < \infty$$

(11)

Lemma. Suppose that the function

$$u(t,x) \in C^{[t]}_{2m,B_{\beta(l)}}(\Pi_t) \quad .$$

$0 \le t \le T.$

Then for $0 < \varepsilon < 1$ the following inequality holds

$$|u|^{[t]}_{2m-1,B_{\beta(l)}} \le \varepsilon |u|^{[t]}_{2m,B_{\beta(l)}} + $$

$$+ C(\varepsilon)|u|^{[t]}_{0,0}$$

For the proof of this lemma see [7].

For equation (4) we consider in addition to the parabolicity condition (5) that there exists a domain

$$H_M = \{(t,x) \in \Pi_T ; |u| \le M, \cdots$$

$$\ldots |p^r| \le M, 1 \le r \le 2m, M = const\}$$ in

which the function $A(t,x,p^0,p^1,\ldots,p^{2m})$ together with its derivatives with respect to $p_k^r,\ p^r = (\ldots,p_k^r,\ldots),\quad 1 \le r = |k| \le 2m$ up to the second order inclusively is continuous, satisfies the Lipschitz condition with respect to $p^r,\quad 1 \le r = |k| \le 2m$ and the general Hölder condition of exponent $\beta(l), l > 0$, with respect to x and with the constant C_M.

Moreover $A(t,x,0,\ldots 0) \in C^{[T]}_{0,\alpha(l)}(\Pi_T)$,

$|A(t,x,0,\ldots 0)|^{[T]}_{0,\alpha(l)} \le C$.

All the mentioned derivatives are bounded in H_M by the constant C_M. Now we shall consider the equation (1) with the initial zero condition:

$$u|_{t=0} = 0 \qquad (12)$$

3. Bounds for solutions to the Cauchy problem for linear parabolic equation

Theorem 1. Let $u(t,x) \in C^{[T]}_{2m,B_{\beta(l)}}(\Pi_T)$ be a solution to the problem (1), (12) in the layer Π_T. Assume that

$$f = f_1 + f_2,\ f_1 \in C^{[t]}_{0,\beta(l)}(\Pi_T),$$

$f_2 \in C^{[t]}_{0,\alpha(l)}(\Pi_T),\ \beta(l), \alpha(l) \in \Gamma$,

$\beta(l) \to \sigma_1,\qquad \alpha(l) \to \sigma_2$

if $l \to 0^+$ **or** $l \to +\infty,\quad 0 \le \sigma_1 < \sigma_2 < 1$.

Furthermore the conditions (3) and (11) hold. Then there exists a constant **K**, depending only on $n, m, \lambda, B, T, \alpha(l), \beta(l)$ *and* $B_{\beta(l)}$ such that for $0 \le t \le T$

$$|u|^{[t]}_{2m,B_{\beta(l)}} \le K[\,|f_1|^{[t]}_{0,\beta(l)} + t^{\frac{\sigma_2-\sigma_1}{2m}} |f_2|^{[t]}_{0,\alpha(l)}] \qquad (13)$$

Remark. We can reduce the Cauchy problem with non-zero initial condition $u|_{t=0} = u_0(x)$ to the Cauchy problem (1) , (12) by means of the transformation $\bar{u} = u(t,x) - u_0(x)$ where

$u_0(x) \in C^{[T]}_{2m,\beta(l)}(E_n)$.

Now we shall establish an estimate of the modulus of continuity with respect to the time t of the derivatives $D_x^k u,\quad |k| \le 2m$ for the solutions to the equation (1).

Theorem 2. Let $u(t,x) \in C^{[T]}_{2m,B_{\beta(l)}}(\Pi_T)$ be a solution to the equation (1) in the cylindrical domain $Q_T = [0,T] \times \Omega,\quad \Omega -$ bounded domain in E_n. Assume that

$f \in C^{[t]}_{0,\beta(l)}(Q_T),\ \beta(l) \in \Gamma$,

$\beta(l) \to \sigma_1,$ **if** $l \to 0^+$ **or** $l \to +\infty,$ $0 \le \sigma_1 < 1$.

Furthermore the conditions (3) and (11) hold. Then if the points ,

$(t_1, x_0), (t_2, x_0) \in Q_T^\delta = (\delta^2, T] \times \Omega^\delta$

$\delta^2 < t_2 < t_1 < T$,

$\Omega^\delta = \{x \in \Omega; dist(x, \partial\Omega) > \delta\}$,there exists a constant **K**, depending only on $n,m,\lambda,B,T,\alpha(l),\beta(l)$ *and* $B_{\beta(l)}$ such that for any derivative $D_x^k u,\quad |k| \le 2m$ of the function $u(t,x)$ the following estimate holds for $j = 0,1,...,2m-1$

$$\frac{\left|D_x^{2m-j}u(t_1,x_0)-D_x^{2m-j}u(t_2,x_0)\right|}{\left(|t_1-t_2|^{\frac{1}{2m}}\right)^{B\beta(|t_1-t_2|^{\frac{1}{2m}})+j}} \le$$

$$\le K[\,|f|_{0,\beta(l)}^{[t_1]} + |u|_{0,0}^{[t_1]}\,]$$

The proof of the theorem 2 is similar to the proof of the theorem 2 in [16] but reasoning as in the proof of theorem 1 of the present paper.

Theorem 3.Suppose that all assumptions of the theorem 1 hold. Then for every $(t_1, x), (t_2, x) \in \Pi_T$, $0 \le t_2 < t_1 \le T$, there exists a constant **K**, depending only on $n,m,\lambda,B,T,\alpha(l),\beta(l)$ *and* $B_{\beta(l)}$, such that for any derivative $D_x^k u,\quad |k| \le 2m$ of the solution $u(t,x)$ to the Cauchy problem (1),(12)the following estimates hold for $j = 0,1,...,2m-1$

$$|u(t_1,x)-u(t_2,x)| \le K\ M|t_1-t_2|$$

(14)

$$|D_x^{2m-j}u(t_1,x)-D_x^{2m-j}u(t_2,x)| \le$$

$$\le K\,M(|t_1-t_2|^{\frac{1}{2m}})^{B\beta(|t_1-t_2|^{\frac{1}{2m}})+j}$$

(15)

where

$$M = [\,|f_1|_{0,\beta(l)}^{[t]} + t_1^{\frac{\sigma_2-\sigma_1}{2m}}\ |f_2|_{0,\alpha(l)}^{[t]}\,]$$

The proof of this theorem is similar to the proof of the theorem 3 in [16] but reasoning as in the proof of theorem 1 of the present paper

4. Existence and uniqueness theorems.

Theorem 4.Suppose that all conditions of theorem 1 are true. Then there exits a unique solution $u(t,x) \in C^{[T]}_{2m,B_{\beta(l)}}(\Pi_T)$ to the Cauchy problem (1) , (2) with continuous derivatives u_t in Π_T.

We can get the proof of this theorem on the basis of the new priori estimates established in this work and with the aid of the method of continuity in a parameter. (see [4] and [20]). We proceed now to formulate the local existence theorem for solutions to the non-linear problems for the equation (4). Here we consider that the function

$$A(t,x,u,D_x u,...,D_x^{2m}u) =$$
$$L(u) + F(t,x,u,D_x u,...,D_x^{2m}u) +$$
$$+ A(t,x,0,...,0)$$

where

$$L(u) = A(t,x,0,...,0)u +$$
$$(A_{p^1}(t,x,0,...,0), D_x u) + ... +$$
$$+ (A_{p^{2m}}(t,x,0,...,0), D_x^{2m}u)$$

Theorem 5. Suppose that all assumptions with respect to the function $A(t,x,p^0,p^1,...,p^{2m})$ hold. Moreover $0 \le \sigma_1 < \sigma_2 < 1$ .Then there exists t_0 determined by the above assumptions, such that the problem (4) ,(12) has in the layer $\Pi_{t_0} = [0,t_0] \times E_n$ a unique solution $u(t,x) \in C^{[t_0]}_{2m,B_{\beta(l)}}(\Pi_{t_0})$ with a continuous derivative u_t in $\Pi_{t_0} = [0,t_0] \times E_n$.

Remark. We can reduce the Cauchy problem with non-zero initial condition $u|_{t=0} = \varphi(x)$ to the Cauchy problem with the zero initial condition $u|_{t=0} = 0$ by means of the transformation $\bar{u} = u(t,x) - \varphi(x)$,

where $\varphi(x) \in C^{[T]}_{2m,\beta(l)}(E_n)$.

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