# Analytical Modelling of Transient Phenomena in PWM Control Circuits

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*Abstract:* - The paper brings a new mathematical model for analysis of circuits containing periodically operated switches. The model uses the Laplace and modified Z transform (mixed p-z approach).This approach enables us to determine the steady-state and transient components of the load current in relatively simple and lucid form. The solution is a closed-form and is not dependent on the number of the pulses of the PWM pattern.

*Key-Words:*-Analytical model, transient phenomena, PWM control, electrical circuits

## 1 Introduction

Several methods have been describes for the analysis of linear circuits containing periodically operated switches [1] [2] [3].However, the approach used in these methods depends heavily on matrix manipulations as they require matrix inversion as well as exponentiation.

In order to satisfy the required conditions for differential state equations describing the circuit behavior the continuity conditions due to the steady-state current at the transitions of states are commonly used ,i.e.:

$$I\,(i_{ts}^{+}) = I\,(i_{ts}^{-}) \tag{1a}$$

also, condition of periodicity must be used:

$$i(0^{+}) = i(T^{-}) \tag{1b}$$

where $t_s$ are switching instants is a period.

Using (1a) and (1b) for the whole period of PWM we get algebraic equations that must be solved to obtain *steady-state* solution In case of *transient* solution when current is not periodic during T we must use (1a) and (1b) for the whole transient duration. The solution of the current response is intricate, as the number of the pulses is increasing. This paper brings a new mathematical model that uses the Laplace and modified Z transform (mixed p-z approach). This approach enables us to determine the steady-state and transient components of the load current in relatively simple and lucid form. The solution is a closed-form and is not dependent on the number of the pulses of the PWM pattern.

## 2 Laplace Transform

### 2.1 Laplace Transform of Voltage Waveform

The energy conversion of many power electronics converters is achieved by cyclically controlled switching topological configurations. Let us consider power electronic circuit with an output voltage of the form shown in Fig.1. If the period of the modulation is T, then the converter generates an input voltage for the system that is typically:

$$u(t) = \begin{cases} V_{dc} \ \text{for} \quad nT + T_{kA} \le t < nT + T_{kB} \\ 0 \ \text{for} \quad nT + T_{kB} \le t < nT + T_{(k+1)A} \end{cases} \tag{1c}$$

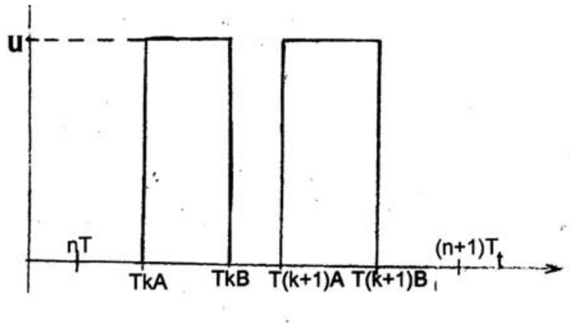


Fig.1Voltage pulse pattern.

Let us express time as follows:

$t=(n+\varepsilon)T$ , $n=0,1,2,...$ , $0<\varepsilon\leq 1$ (2)

then (1c) can be expressed in per unit time

$$v(t) = \begin{cases} V_{dc} \text{ for } \varepsilon_{kA}\leq\varepsilon<\varepsilon_{kB} \\ 0 \text{ for } \varepsilon_{kB}\leq\varepsilon<\varepsilon_{(k+1)A} \end{cases} \quad (3)$$

The traditional PWM waveform for the full-bridge inverter is shown in Fig.2.The output voltage changes between +U and zero or between –U and zero.

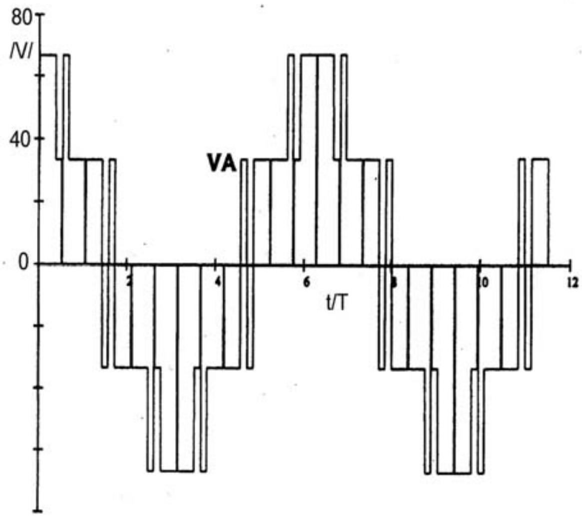


Fig.2 PWM pattern of full-bridge voltage inverter

Using the definition of the Laplace transform for periodic signal we can write:

$$U(p) = \frac{\int_0^{2T} u(t)e^{-pt}dt}{1-e^{-2pT}} = \frac{U}{p}\frac{e^{pT}}{(1+e^{pT})} \quad (4)$$

$$\sum_{k=1}^{M}(e^{-pT\varepsilon_{kA}} - e^{-pT\varepsilon_{kB}})$$

where

$\varepsilon_{kA}T, \varepsilon_{kB}T$ are the beginning and end of k-th pulse, respectively.M is number of pulses within period T.

Now, we suppose this voltage waveform is feeding a static inductive load with impedance

$$Z(p)=R + pL \quad (5)$$

The Laplace transform of the load current is given by

$$I(p) = \frac{U(p)}{Z(p)} =$$

$$= \frac{U}{p}\frac{e^{pT}}{(1+e^{pT})(R+pL)}\sum_{k=1}^{M}(e^{-pT\varepsilon_{kA}} - e^{-pT\varepsilon_{kB}}) \quad (6)$$

## 2.2 Z-transform of Load Current

As an equation (6) contains infinite number of poles given by

$$1+e^{pT} = 0 \quad (7)$$

we can not use the inverse Laplace transform directly. But (7) can be evaluated in closed form using the modified Z transform [4].The modified Z transform is defined by the infinite sequence as follows:

$$X(z,\varepsilon) = \sum_{i=0}^{\infty}\{ x(i+\varepsilon)T.z^{-i} \}, \quad z = e^{pT} \quad (8)$$

next, time is expressed as

As can be seen (6) contains two variables, namely p and $e^{pT}$, and it can be written in a multiplicity form

$$I(p, e^{pT}) = R(e^{pT})Q(p) \quad (9)$$

comparing (9) with (4) we get for the polynomials R and Q

$$R(e^{pT}) = \frac{e^{pT}}{1+e^{pT}}, \quad (10)$$

$$Q(p) = \frac{U}{p(R+pL)}\sum_{k=1}^{M}(e^{-pT\varepsilon_{kA}} - e^{-pT\varepsilon_{kB}})$$

A multiplicative form of (9) can be transferred into modified Z transform. If doing so, we get:

$$I(z,\varepsilon) = R(z) Z_m \{Q(p)\} \tag{11}$$

where

$Z_m$ means transfer from the Laplace into modified Z transform.

In order to find Z transform of Q(p) we must use the translation theorem in Z transform which holds:

$$Z_m\{e^{-p.a}.F(p)\} = z^{-x}.F(z,\varepsilon-a+x) \tag{12a}$$

where parameter x is given by

$$x = \begin{cases} 1 & \text{for } 0 \le \varepsilon < a \\ 0 & \text{for } a \le \varepsilon < 1 \end{cases} \tag{12b}$$

If we want to express translation for k-th pulse, with the beginning $\varepsilon_{kA}$ and the end $\varepsilon_{kB}$ ,(pulse-width $\Delta\varepsilon_k = \varepsilon_{kB} - \varepsilon_{kA}$ ), we can use two parameters, namely $m_k$ and $n_k$ to determine per unit time for prepulse,inside-pulse and postpulse instances, respectively.

$m_k$ is a parameter that defines the beginning of the pulse $\varepsilon_{kA}$, $n_k$ is a parameter that defines the end of the pulse $\varepsilon_{kB}$ According to (12) we can write:

$$m_k = \begin{cases} 1 & \text{for } 0 \le \varepsilon < \varepsilon_{kA} \\ 0 & \text{for } \varepsilon_{kA} \le \varepsilon < 1 \end{cases} \qquad n_k = \begin{cases} 1 & \text{for } 0 \le \varepsilon < \varepsilon_{kB} \\ 0 & \text{for } \varepsilon_{kB} \le \varepsilon < 1 \end{cases} \tag{13}$$

By means of $m_k$ and $n_k$ we can express per unit time for the three intervals :

a) $0 < \varepsilon \le \varepsilon_{kA}$ prepulse per unit time. $m_k=1, n_k=1$
b) $\varepsilon_{kA} < \varepsilon \le \varepsilon_{kB}$ inside pulse per unit time .$m_k=0, n_k=1$
c) $\varepsilon_{kB} < \varepsilon \le 1$ postpulse per unit time .$m_k=0, n_k=0$

According to the residual theorem we can find :

$$L^{-1}\left\{\frac{1}{p(R+pL)}\right\} = \frac{1}{R} - \frac{1}{R} e^{-\frac{R}{L}t} \tag{14}$$

$L^{-1}$ is a symbol for inverse Laplace transform.

Using (14)(13) and (10) we can find Z transform of (9) as follows:

$$I(z,\varepsilon) = \\ = \frac{U}{R}\left\{ \frac{z}{z+1}\left\langle \frac{z}{z-1}(z^{-m_k} - z^{-n_k}) + \right.\right. \\ + \frac{ze^{-\frac{RT}{L}\varepsilon}}{z - e^{-\frac{R}{L}T}}\left[ z^{-m_k} e^{-\frac{R}{L}T(m_k - \varepsilon_{kA})} \right. \\ \left.\left.\left. - z^{-n_k} e^{-\frac{RT}{L}(n_k - \varepsilon_{kB})} \right\rangle\right\}\right] \tag{15}$$

2.3 Original Function –Time-Domain Analysis

Now, we can find the original of (15) by the definition of the inverse Z transform:

$$I(n,\varepsilon) = \frac{1}{2\pi j}\oint I(z,\varepsilon) z^{n-1} dz \tag{16}$$

An integral (16) may be solved by means of the residual theorem.

In the next part, we find the inverse of (15) as follows:

Equation (15) has three simple poles namely:

$z_1=1, z_2=-1, z_3=e^{-RT/L}$

a) pole $z_1=1$
as is valid for every k

$$\left[(1)^{-m_k} - (1)^{n_k}\right] = 0 \tag{16}$$

This pole does not give any current component
b) pole $z_2=-1$
this pole gives two current components:

$$i_{S1}(n,\varepsilon) = \frac{U}{R}\frac{(-1)^n}{2}\sum_{k=1}^{M}\left[(-1)^{-m_k} - (-1)^{-n_k}\right] \tag{17}$$

$$i_{S2}(n,\varepsilon) = \frac{-U}{R}\frac{(-1)^n}{(1+e^{-\frac{RT}{L}})}\sum_{k=1}^{M}\left[\begin{matrix}(-1)^{-m_k} e^{-\frac{RT}{L}(m_k - \varepsilon_{kA})} - \\ -(-1)^{-n_k} e^{-\frac{RT}{L}(n_k - \varepsilon_{kB})}\end{matrix}\right].e^{-\frac{RT}{L}\varepsilon} \tag{18}$$

Both terms in (18) form the steady-state components of the load current.

$$i_S(n,\varepsilon) = i_{S1}(n,\varepsilon) + i_{S2}(n,\varepsilon) \tag{19}$$

As an example Fig.3 shows $i_{S1}$ versus $t=(n+\varepsilon)T$,and Fig.4 shows $i_{S2}$ again in time dependency for PWM pattern with:

$M=2, \varepsilon_{1A}=0, \varepsilon_{1B}=0.6, \varepsilon_{2A}=0.8, \varepsilon_{2B}=0.85,$

The parameters of the load are as follows: $R=0.85\Omega, L=0.011H, U=10$ V, $\omega=314 s^{-1}, T=\pi/\omega=0.01$ s.

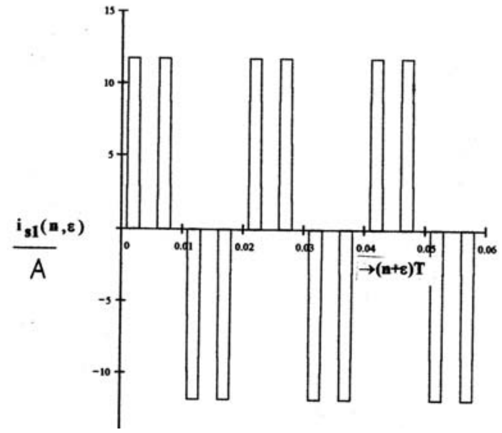


Fig.3 First part $i_{S1}(n,\varepsilon)$, forming steady-state current component

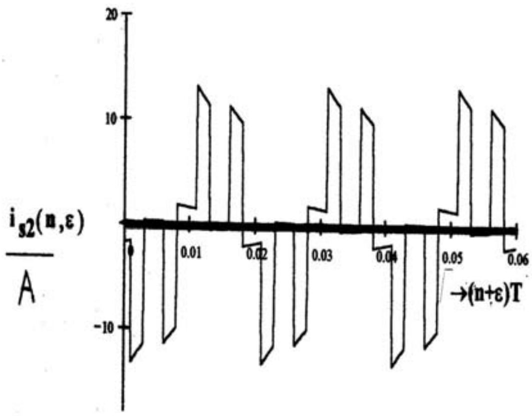


Fig.4 Second part $i_{S2}(n,\varepsilon)$ forming steady-state component

c) pole $z_3=e^{-RT/L}$

Using the residua theorem we get again:

$$i_T(n,\varepsilon)=\frac{U}{R}\frac{e^{-\frac{RT}{L}}}{1+e^{-\frac{RT}{L}}}\sum_{k=1}^{M}\left[e^{\frac{RT}{L}\varepsilon_{kB}}-e^{\frac{RT}{L}\varepsilon_{kA}}\right]e^{-\frac{RT(n+\varepsilon)}{L}} \quad (20)$$

It may be proved that:

$$\lim_{n\to\infty} i_T(n+\varepsilon)=0$$

this term forms the transient component, vanishing for $n\to\infty$.

Fig.5. shows time dependency of transient component $i_T$ for the same parameters as in previous figures.

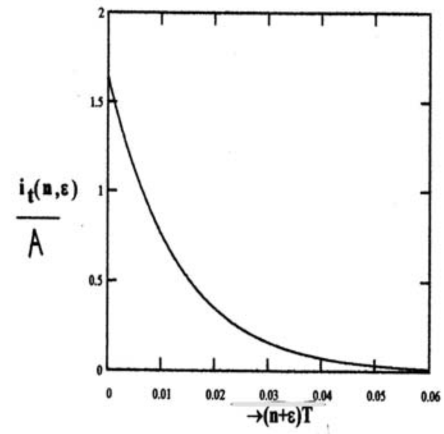


Fig.5 Transient component of the load current for PWM pattern

The overall load current can be expressed as the sum of all terms:

$$i(n,\varepsilon)=i_{S1}(n,\varepsilon)+i_{S2}(n,\varepsilon)+i_T(n,\varepsilon) \quad (21)$$

The overall load current is shown in Fig.6

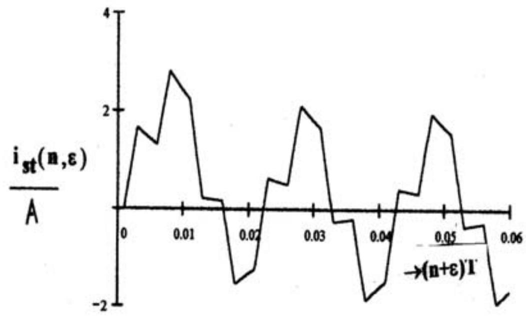


Fig.6. Overall load current for PWM pattern

As can be seen all terms in (21) have clear mathematical and physical meaning. The first term has the same shape as the PWM voltage waveform. The second one is composed from exponential curves opposite to the firs term. Both terms have a big magnitude. The last term with lower magnitude forms the transient component of the load current. Transient component is not dependent on PWM pattern.

From equation for PWM pattern we can easily derived square waveform (without PWM) .
We substitute in (21) $M=1, \varepsilon_{1A}=0$, $, \varepsilon_{1B}=1$, $m_1=0, n_1=1$ and from (21) we obtain:

$$i_S(n,\varepsilon) = \frac{U}{R}\left[\begin{array}{l} (-1)^n - 2\dfrac{(-1)^n}{1+e^{-RT/L}} e^{-\frac{RT}{L}\varepsilon} + \\ + \dfrac{(1-e^{-RT/L})}{(1+e^{-RT/L})} e^{-\frac{RT}{L}(n+\varepsilon)} \end{array}\right] =$$
$$= i_{SS1}(n,\varepsilon) + i_{SS2}(n,\varepsilon) + i_{ST}(n,\varepsilon) \qquad (22)$$

This equation was also derived in [5],but by more complicated way.Again,there are two terms ($i_{SS1}$ and $i_{SS2}$) forming the steady-state component.$i_{SS1}$ is formed by square waveform,$i_{SS2}$ is formed by exponential curves.$i_{ST}$ is a transient component vanishing for $t \to \infty$.
Fig.7 shows the overall load current for square voltage waveform (without PWM pattern)

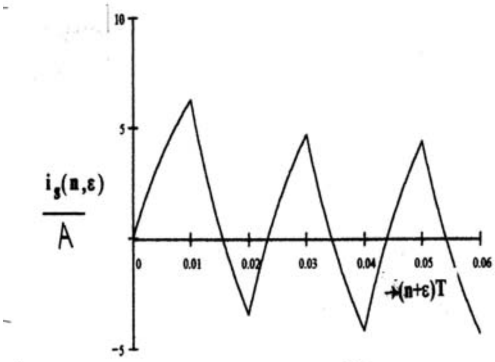


Fig.7 Overall load current for square voltage waveform(without PWM pattern)

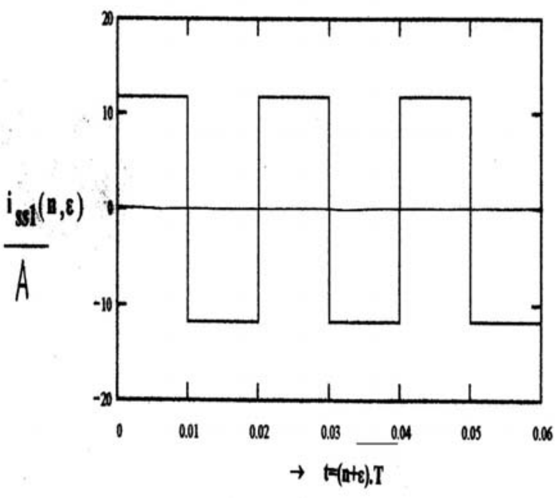


Fig.8 First part forming steady-state load current (without PWM pattern)

From Figrs.8 and 9 we can see both components forming the steady-state current.

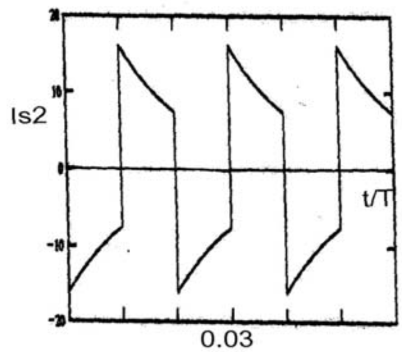


Fig.9 Second part forming steady-state load current(without PWM pattern)

## 3 Conclusion

The paper presents mathematical model for analyzing circuits with PWM waveforms. The expressions which are derived have clear mathematical and physical meaning. The model uses mixed p-z approach that enables us to determine both steady-state and transient components of the load current to be calculated directly under PWM pattern. The change of switching instants is reflected in the solution by a change in the values $m_k$ and $n_a$.The proposed method can be also used for another types of periodic chopped waveforms such as DC-DC converters, three phase inverters etc.

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