Robust Optimal Control in Not-Completely Controllable Linear Systems

ELLIDA M. KHAZEN
1286 Mason Mill Ct, Herndon, VA 20170
U.S.A.

Abstract: Robust control problems in linear systems are considered on the base of analysis of related linear quadratic differential game. New explicit formulae for the "best" robust optimal control input and the "worst" exogenous disturbance derived with the use of pseudo-inverse matrices $D^{+}(t)$ were originally suggested in this author's previously published paper "Reduction of Dimensionality in Choosing Robust Optimal Control in Linear Systems", WSEAS Transactions on Mathematics, Issue 4, Volume 2, October 2003, pp. 318 - 323. The meaning of the use of the pseudo-inverse matrices is that some uncontrollable or almost uncontrollable components are automatically eliminated in choosing optimal control. In this paper the proof of the main theorem as well as meaningful and detailed analysis are presented. We shall demonstrate that the extremal that corresponds to the saddle point of the game may be extended beyond conjugates points (under assumption that $D(t)$ remains non-negative definite). We reveal that computational difficulties, which may arise in obtaining the solution of the game and the optimal control law, are related to the fact that the trajectories $x(t)$ (where the state $x = (x_1,..., x_n)$) of the system may be exactly or approximately localized in the subspace of dimensionality less than n on some time interval. We shall demonstrate how to overcome these difficulties by using "inverse" matrix differential equations of Riccati type and pseudo-inverse matrices.

Keywords: linear quadratic differential game; linear robust control; matrix Riccati differential equation; pseudo-inverse matrix.

1 Introduction

Let the considered controlled system be described by the linear differential equation

$$\frac{dx(t)}{dt} = A(t)x(t) + B_1(t)w(t) + B_2(t)u(t), \qquad (1)$$

$$x(t_0) = x_0,\ t \geq t_0,\ \ z(t) = C(t)x(t),$$

where the vector function $x(t)$ represents the state of the system, $x = (x_1,..., x_n)$; $u(t)$ is a control input and $w(t)$ is an external disturbance.

A solution to the problem of the optimal robust system design is closely related to a linear quadratic differential game (e.g., see books [1-4] and references therein).

Here we shall consider this problem on the finite time interval, $t_0 \leq t \leq T$ (the finite-horizon case). The solution of the robust optimal control problem in the finite-horizon case is equivalent to solving the following differential game: Find the "best" optimal input $u^*(t)$ and the "worst" exogenous input $w^*(t)$, which provide a saddle point

$$J(u^*, w) \leq J(u^*, w^*) \leq J(u, w^*) \qquad (2)$$

for the functional

$$J(u,w) = \int_{t_0}^{T} \Big[x^T(t)C^T(t)C(t)x(t) + u^T(t)u(t) - \gamma^2 w^T(t)w(t) \Big] dt + x^T(T)\Delta x(T), \qquad (3)$$

where Δ is an arbitrary non-negative definite symmetric matrix, and $x(t)$ satisfies ODE (1).

The solution of the differential game (2), (3) for the system (1) was investigated in [1-4]. It was based on the solution $P(t)$ to the following matrix differential Riccati equation:

$$-\frac{dP(t)}{dt} = A^T P + PA - P(B_2 B_2^{\ T} - \gamma^{-2} B_1 B_1^{\ T})P + C^T C \qquad (4)$$

with the boundary condition $P(T) = \Delta$, $t_0 \leq t \leq T$. The solution of the linear quadratic optimal control problem, which is obtained from (1), (2), (3) by imposing $w(t) \equiv 0$, was also investigated.

As was shown in author's previous publications [5 -7], the computational difficulties, which might arise in solving the Riccati equations for the matrix function $P(t)$ for time-varying systems, are related to the fact that trajectories $x(t)$ may exactly or approximately be localized in the subspace of dimensionality less than n on some time interval. The matrix $D(t)$, defined as a solution to the "inverse" matrix Riccati differential equation

$$\frac{dD(t)}{dt} = AD + DA^T - (B_2 B_2{}^T - \gamma^{-2} B_1 B_1{}^T) + DC^T CD \tag{5}$$

on the time interval $t_0 \le t \le T$, with boundary condition $D(T) = \Delta^{-1}$ or $D(T) = \Delta^+$ if Δ is a singular matrix, and the pseudo-inverse matrix $D^+(t)$ were introduced in [5 -7]. Here Δ^+ and $D^+(t)$ denote the Moore-Penrose pseudo-inverse matrices. It was shown in [5 -7] that the matrix $D(t)$ becomes close-to-singular or $D(t)$ stays singular (i.e. rank deficient, degenerated) all the time on a considered time interval in the cases where the trajectories $x(t)$ approximately or exactly belong to a subspace of dimensionality less than n (on this time interval). The matrix $P(t)$ in such cases may be increasing significantly on this time interval, while the matrix $D(t)$ remains uniformly bounded so that the solution of the "inverse" matrix equation (5) does exists. The work [7] represents the generalization of the results obtained in [5, 6] in the case of linear quadratic optimal control problems (which corresponds to the case $w(t) \equiv 0$), to the general case of robust linear optimal control problems.

For the finite-horizon case, we shall only assume that the matrices $A(t)$, $B_1(t)$, $B_2(t)$, $C(t)$ are piecewise continuous functions with respect to t. In this work we shall prove the main Theorem that provides the sufficient conditions for existence of the saddle point in the differential game (2), (3), (1). We shall demonstrate that the "best" control input and the "worst" exogenous input are determined by the following formulae:

$$u^*(t) = -B_2{}^T(t) D^+(t) x(t), \tag{6}$$

$$w^*(t) = \gamma^{-2} B_1{}^T(t) D^+(t) x(t). \tag{7}$$

With the initial condition $x(t_0) = 0$, the control law (6) provides the robust control.

We want to emphasize that in this work the new solutions to the optimal control problems are found and the new methods are developed with the use of *pseudo-inverse* matrices $D^+(t)$.

In the case $rank(D(t)) = q < n$, the meaning of the expression (6) is that some uncontrollable or almost-uncontrollable components $x'_k(t)$ of the vector $x'(t) = M^{-1}(t)x(t)$ are automatically eliminated in choosing optimal control with help of the pseudo-inverse matrix $D^+(t)$. Here $M(t)$ is a matrix of a linear orthogonal transformation $x(t) = M(t)x'(t)$ [5 -7]. This paper contains the proof of the main theorem and provides new methods.

2 Main Theorem

At first we shall demonstrate in more details that the matrix function $D^+(t)$ does not always coincide with $P(t)$, and the obtained formulae (6), (7) provide indeed the new solutions to the robust optimal control problems in non-completely controllable systems which overcome computational difficulties.

Let $D_1(t)$ represents the solution to the "inverse" matrix equation (5) on the time interval $t_0 \le t \le T$ with the boundary condition $D_1(T) = \Delta^+$, and $P(t)$ satisfies the equation (4) with $P(T) = \Delta$. Let the boundary matrix Δ and the matrices $B_1(t)$, $B_2(t)$ are singular, and the matrices $B_1(t), B_2(t)$ enjoy q_2 linearly independent rows, with $q_2 < n$, but the given matrix $C(t)$ is non-singular at all $t \in [t_0, T]$. Thus, denote $rank(\Delta) = q_1$, $q_1 < n$, and $rank(B_i(t)) \equiv q_2$, $q_2 < n$, $i = 1,2$, while $rank(C(t)) \equiv n$. In [7, Section 3] the orthogonal transformations $x = M(t)x'$, $x' = M^{-1}(t)x$ of the coordinate system were considered. In the new coordinate system x' the matrices $B_i(t)$ ($i = 1,2$) would be transformed to $B'_i(t) = M^{-1}(t)B_i(t)$. In the case of not-completely controllable system, which corresponds to rank deficient matrix $D(t)$ [5-7], there exists the orthogonal transformation $M^{-1}(t)$ such that ($q_2 + 1$)-th,…, n-th rows of the matrices $B'_1(t), B'_2(t)$ become null rows identically on the time interval $t_0 \le t \le T$. Here we suggest that $B_i'(t)$ ($i = 1,2$) enjoy that property.

Consider the solutions of the equations (5) and (4). We assume that the matrix functions $D_1(t)$ and $P(t)$ exist, i.e. remain bounded at all $t \in [t_0, T]$. Consider the solutions backward in time and introduce $\tau = T - t$, $\overline{D}_1(\tau) = D_1(T-t), \overline{P}(\tau) = P(T-t)$. Then we can rewrite the equation (5) in the form

$$\frac{d\overline{D}_1(\tau)}{d\tau} = -(A + \frac{1}{2}\overline{D}_1 C^T C)\overline{D}_1$$
$$- \overline{D}_1(A + \frac{1}{2}\overline{D}_1 C^T C)^T + B_2 B_2^T - \gamma^{-2} B_1 B_1^T$$

with the boundary condition $\overline{D}_1(0) = \Delta^+$.

Similarly to [5-6] we obtain that the solution $\overline{D}_1(\tau)$ can be written in the form

$$\overline{D}_1(\tau) = \Phi_1(\tau,\tau_0)\overline{D}_1(\tau_0)\Phi_1^T(\tau,\tau_0) + \int_{\tau_0}^{\tau} \Phi_1(\tau,s)[B_2(s)B_2^T(s) - \gamma^{-2} B_1(s)B_1^T(s)]\Phi_1^T(\tau,s)ds$$

where $\Phi_1(\tau,\tau_0)$ is the transition matrix of the linear system

$$\frac{dz_1}{d\tau} = -\left(A(\tau) + \frac{1}{2}\overline{D}_1(\tau)C^T(\tau)C(\tau) \right) z_1, \; \tau \ge \tau_0.$$

The matrix $\Phi_1(\tau,\tau_0)$ is non-singular, and $\Phi_1(\tau_0,\tau_0) = I$, where I stands for the unity matrix. In the case at hand, $\tau_0 = 0$, and $\overline{D}_1(\tau_0) = \Delta^+$. The matrix Δ can be chosen in such way that $rank(D_1(t)) \equiv \max\{q_1, q_2\} < n$.

Now consider the solution $\overline{P}(\tau)$ of the equation (4) (considered backward in time) with the boundary condition $\overline{P}(0) = \Delta$. Similarly, we obtain

$$\overline{P}(\tau) = \Phi_2(\tau,\tau_0)\overline{P}(\tau_0)\Phi_2^T(\tau,\tau_0) + \int_{\tau_0}^{\tau} \Phi_2(\tau,s)C^T(s)C(s)\Phi_2^T(\tau,s)ds, \qquad (8)$$

where $\Phi_2(\tau,\tau_0)$ is the transition matrix of the linear system

$$\frac{dz_2}{d\tau} = \{A^T(\tau) - \frac{1}{2}\overline{P}(\tau)[B_2(\tau)B_2^T(\tau) - \gamma^{-2} B_1(\tau)B_1^T(\tau)]\} z_2$$

with $\tau \ge \tau_0$.

Hence we obtain that $rank(P(t)) \equiv rank(C(t)) \equiv n$ at all $t_0 \le t < T$, but $rank(P(t))$ decreases at $t = T$, since $rank(P(T)) = rank(\Delta) = q_1 < n$.

It follows that $P(t)$ cannot coincide with $D_1^+(t)$, since $rank(D_1^+(t)) = rank(D_1(t)) < rank(P(t))$.

If $t_0 \le t < T$, there exist the inverse matrix $P^{-1}(t)$, which satisfies the "inverse" equation (5). Denote that solution $D_2(t) = P^{-1}(t)$. As $t \to T$, $D_2(t)$ becomes unbounded. Yet $D_2^+(t) = P(t) \to \Delta$ as $t \to T$ and $t < T$. Thus, we discover that the solution $D(t)$ of the equation (5) (on the time interval $t_0 \le t < T$) that satisfies the boundary condition $D^+(t) \to \Delta$ as $t \to T$ is not unique: we obtain the solution $D_2(t)$ as well as $D_1(t)$.

We shall consider below the Euler system of differential equations (9), (10), which determine extremals $x^*(t)$, $\lambda^*(t)$ of the functional J (3). The saddle point can only be reached on the extremal. It is known that if the solution $P(t)$ of the equation (4) exists then along the extremal the linear relation holds: $\lambda^*(t) = P(t)x^*(t)$. Further we shall consider the case when the initial vector $x(t_0)$ allows the representation of the form $x(t_0) = D(t_0)y_0$. We shall then prove that along the extremal another linear relation also holds: $x^*(t) = D(t)\lambda^*(t)$, where $D(t)$ satisfies the "inverse" equation (5) with the boundary condition $D^+(T) = \Delta$ at $t = T$. In the case at hand, we obtain $x^*(t) = P^{-1}(t)\lambda^*(t) = D_2(t)\lambda^*(t)$ (when $t_0 \le t < T$) as well as $x^*(t) = D_1(t)\lambda^*(t)$. But the solution of the Euler system of differential equations (9), (10) with given bounded terminal vectors $x(T)$ and $\lambda(T) = \Delta x(T)$ is unique. That means that the extremal may be constructed with the use of

$D_1(t)$ as well as with the use of $P(t)$, if the initial vector $x(t_0)$ allows the representation of the form $x(t_0) = D(t_0)y_0$. In general case, the terminal vector $x(T)$ ought to allow the representation $x(T) = \Delta^+\lambda(T) + q(T)$ where $\Delta q(T) = 0$, so that the boundary condition $\Delta x(T) = \lambda(T)$ be satisfied. With the use of $P(t)$ the solution of the Euler differential equations (9), (10) with the terminal conditions $x(T)$, $\lambda(T) = \Delta x(T)$ and $x(T) = \Delta^+\lambda(T)$ would be presented in the form $x(t)$, $\lambda(t) = P(t)x(t)$. With the use of $D_1(t)$ the same extremal would be presented in the form $x(t) = D_1(t)\lambda(t)$, $\lambda(t)$ (so that the initial vector $x(t_0) = D_1(t_0)\lambda(t_0)$). But the use of the pseudo-inverse matrix ${D_1}^+(t)$ provides opportunity to avoid computational difficulties that might arise in obtaining the matrix function $P(t)$, since $\overline{P}(\tau)$ might be increasing significantly as τ and T increases (as shown in [7, Section 3]), while $\overline{D}_1(\tau)$ and $\overline{D}_1{}^+(\tau)$ remains uniformly bounded. Besides, it might even occur that $\overline{P}(\tau)$ becomes unbounded, i.e. the solution $P(t)$ does not exist on the whole time interval $t_0 \le t \le T$, while the non-negative definite solution $D_1(t)$ still exists.

Furthermore, with the use of the pseudo-inverse matrix ${D_1}^+(t)$ the uncontrollable or almost uncontrollable components may be automatically eliminated in choosing optimal control. Although the "best choice" of the boundary matrix Δ, that excluded uncontrollable or almost uncontrollable components, might be unknown in advance, it could be found with aid of computation of $\overline{D}_1(\tau)$, and the robust control could be determined with the use of $\overline{D}_1{}^+(\tau)$.

The consideration given in [7, Section 3] also confirms that with the use of the pseudo-inverse matrix $\overline{D}_1{}^+(\tau)$ the uncontrollable components may be automatically eliminated in choosing optimal control, and the computational difficulties may be overcome.

Theorem. (Sufficient conditions). For the saddle point (2) of the functional (3) subject to condition (1) to exist, it is sufficient that the following conditions are satisfied:

(a) there exists the non-negative definite solution $D(t)$ (in other words, the guarantee that $D(t)$ remains bounded and non-negative definite) to the matrix differential equation (5), on the time interval $t_0 \le t \le T$, with the boundary condition $D(T) = \Delta^+$, where Δ is the non-negative definite symmetric matrix introduced in (3);

(b) the initial condition $x(t_0) = x_0$ allows representation of the form $x_0 = D(t_0)\lambda_0$.

Then the "best" control law and the "worst" external disturbance are determined by the formulae (6) and (7), that correspond to this saddle point.

Proof. Suppose that there exists the solution $D(t)$ of the equation (5) on the interval $t_0 \le t \le T$ (i.e. $D(t)$ remains bounded) with a boundary condition $D(T) = \Delta^+$, and $D(t)$ is non-negative definite. We shall prove then that there exists a saddle point (2) of functional (3) provided that $x(t)$ satisfies the equation (1) with the initial vector $x(t_0) = x_0$ that satisfies the condition (b). The "best" control input $u^*(t)$ and the "worst" disturbance $w^*(t)$, defined by (6) and (7), shall correspond to this saddle point.

For the saddle point (2), (3), (1) to exist it is necessary that the first variance of the functional (3) subject to condition (1) vanishes at that saddle point. Then on the basis of standard procedure from calculus of variations we obtain the known Euler differential equations:

$$\frac{dx^*}{dt} = Ax^* - (B_2{B_2}^T - \gamma^{-2}B_1{B_1}^T)\lambda^*, \qquad (9)$$

$$\frac{d\lambda^*}{dt} = -C^TCx^* - A^T\lambda^*, \qquad (10)$$

$$u^* = -{B_2}^T\lambda^*, \qquad (11)$$

$$w^* = \gamma^{-2}{B_1}^T\lambda^*, \qquad (12)$$

with boundary conditions

$$x^*(t_0) = x_0, \ \lambda^*(T) = \Delta x^*(T). \qquad (13)$$

The solution that satisfies (9), (10) and (13) is referred to as an extremal. The saddle point of the functional (3) can only be reached on an extremal.

We can check directly, substituting $x(t) = D(t)\lambda(t)$ into (9), (10), that the pair of functions $x(t) = D(t)\lambda(t)$, $\lambda(t)$ satisfies the Euler equations (9), (10) with initial conditions $x(t_0) = x_0$, $\lambda(t_0) = \lambda_0$, if $x(t_0) = D(t_0)\lambda(t_0)$.

Choose $\lambda_0 = D^+(t_0)x(t_0)$, then the linear relation $\lambda(t) = D^+(t)x(t)$ holds along that trajectory. Then the boundary condition $\lambda(T) = \Delta x(T)$ is satisfied, so that we obtain an extremal.

We shall demonstrate now that the obtained extremal does indeed provide a saddle point for the functional J. At first we consider the case when $D(t)$ varies without changing rank on the considered time interval $t_0 \le t \le T$ .The proof is done via the following calculation similar to the known method of "completing the square". We can rewrite (3) as

$$J = x^T(t_0)D^+(t_0)x(t_0) + \int_{t_0}^{T} \{x^T(t)C^T(t)C^T(t)x(t) + u^T(t)u(t)\}dt - \int_{t_0}^{T} \{\gamma^2 w^T(t)w(t) + \frac{d}{dt}(x^T(t)D^+(t)x(t))\}dt$$

A pseudo-inverse matrix $D^+(t)$ of any symmetric matrix $D(t)$ that varies without changing rank satisfies [6, 8]

$$\frac{dD^+(t)}{dt} = -D^+ \frac{dD}{dt} D^+. \qquad (14)$$

We shall need the following property of any solution of equation (1) with the boundary condition $x(t_0) = x_0 = D(t_0)\lambda_0$: the solution $x(t)$ can be presented as

$$x(t) = D(t)y(t). \qquad (15)$$

Indeed, the function $y(t)$ can be obtained as a solution to the following ODE:

$$\frac{dy}{dt} = -D^+DA^T y + D^+(B_2 B_2{}^T - \gamma^{-2}B_1B_1{}^T)y - D^+DC^T CDy + D^+B_1 w(t) + D^+B_2 u(t),$$

with the initial condition $y(t_0) = \lambda_0 = D^+(t_0)x(t_0)$, $t_0 \le t \le T$.

Using (14), (15) and (1), we obtain

$$J = x^T(t_0)D^+(t_0)x(t_0) + \qquad (16)$$
$$\int_{t_0}^{T} [(u(t) + B_2{}^T(t)D^+(t)x(t))^T (u(t) + B_2{}^T(t)D^+(t)x(t)) - \gamma^2(w(t) - \gamma^{-2}B_1{}^T(t)D^+(t)x(t))^T (w(t) - \gamma^{-2}B_1{}^T(t)D^+(t)x(t))]dt$$

Recall, that $x^T(t_0)D^+(t_0)x(t_0) \ge 0$ due to the assumption of the Theorem.

The expression (16) shows that the functional J does indeed have a saddle point which is achieved with the "best" control input $u^*(t) = -B_2{}^T(t)D^+(t)x(t)$ and the "worst" exogenous input $w^*(t) = \gamma^{-2}B_1{}^T(t)D^+(t)x(t)$. It also shows that for the initial condition $x(t_0) = 0$, the value of the functional J (3) at the saddle point is zero. Since $x^T(T)\Delta x(T) \ge 0$, the robust control law is defined as a feedback (6).

Now consider the hypothetical case when $rank(D(t))$ could decrease at some moments $t_k{}^*$, $k = 1,..,r$, $T > t_r{}^* > ... > t_1{}^* > t_0$. We consider the evolution of $D(t)$ backward in time. It suffice to consider the evolution on the first time interval $T \ge t \ge t_r{}^*$, because the other cases are similar. We have assumed that the initial value $x(t_0)$ allows the representation of the form $x(t_0) = D(t_0)\lambda(t_0)$. Consider the solution of the Euler linear equations (9), (10) with the initial conditions $x(t_0) = x_0$, $\lambda(t_0) = D^+(t_0)x_0$. Denote that solution $x^*(t)$, $\lambda^*(t)$. Since $D(t)$ satisfies (5) and $x(t_0) = D(t_0)\lambda(t_0)$, $\lambda(t_0) = D^+(t_0)x_0$, along the trajectory the following relations hold:

$$x^*(t) = D(t)\lambda^*(t), \qquad (17)$$

$$\lambda^*(t) = D^+(t)x^*(t). \qquad (18)$$

The extremal $x^*(t)$, $\lambda^*(t)$ remains bounded and differentiable with respect to t on the finite time interval $t_0 \le t \le T$ and satisfies the boundary conditions $x^*(t_0) = x_0$, $\lambda^*(T) = \Delta x^*(T)$.

Denote the functional J (3) as $J[t_0, x(t_0)]$ to indicate explicitly the dependence on the initial condition of the trajectory $x(t)$. For $t_r{}^* < t \le T$ we obtain, along the extremal

$$J[t, x^*(t)] = x^{*^T}(t) D^+(t) x^*(t) = \lambda^{*^T}(t) D(t) \lambda^*(t) \quad (19)$$

and as $t \to t_r^*$ the value (19) remains bounded and tends to the finite limit, with $x^*(t_r^*)$ and $\lambda^*(t_r^*)$ being substituted instead of $x^*(t)$ and $\lambda^*(t)$ into the right-hand side of the formula (19). Furthermore, the function $x^{*^T}(t) D^+(t) x^*(t)$ = $\lambda^{*^T}(t) D(t) \lambda^*(t)$ remains bounded and differentiable with respect to t on the whole time interval $t_0 \le t \le T$.

Note, that even when the matrix $D^+(t)$ becomes unbounded as $t \to t_r^*$ (with $t > t_r^*$) the limit of the value $x^{*T}(t) D^+(t) x^{*T}(t)$ remains bounded and becomes equal to $x^{*T}(t_r^*) \hat{D}^+(t_r^*) x^*(t_r^*)$ where $\hat{D}^+(t_r^*)$ represents the pseudo-inverse matrix of the matrix $\hat{D}(t_r^*) = \lim\limits_{t \to t_r^*} D(t)$, with $rank(\hat{D}(t_r^*)) < rank(D(t))$ at $t > t_r^*$.

If $t \in (t_k^*, t_{k+1}^*)$, with $k = 0,1,..,r$ (where we denote $t_0^* = t_0$, $t_{r+1}^* = T$) the piece of the extremal $x^*(t)$, $\lambda^*(t)$ on each of that time intervals provide the "optimal", mini-maximum value (2) to the functional of the form (3) considered on this interval, with obtained terminal matrix $\Delta_k = \hat{D}^+(t_{k+1}^*)$ at $t = t_{k+1}^*$ and the initial condition $x(t) = x^*(t_k^*)$ at $t = t_k^*$.

Then the expression (19) holds for the functional (3) on the extremal $x^*(t)$, $\lambda^*(t)$. The value of the functional (3) on that extremal equals to

$$J^*[t_0, x(t_0)] = x_0^T D^+(t_0) x_0 = \lambda_0^T D(t_0) \lambda_0 .$$

The mini-maximum value of the functional (3) could be only achieved on an extremal, since that condition is necessary. We have found the extremal $x^*(t)$, $\lambda^*(t)$, and if the control input $u(t)$ would be different from the "optimal" control law $u^*(t) = -B_2^T(t) D^+(t) x(t)$ on some time interval then the value of the functional (3) would not be lesser Similarly, $w^*(t) = -B_1^T(t) D^+(t) x(t)$ represent the "worst" disturbance. Hence, the extremal $x^*(t)$, $\lambda^*(t)$ provides the mini-maximum value of the functional J (3), with the "best" control law (6) and the "worst" law of disturbance (7).

The proof of the Theorem is completed.

4 Conclusion

In this work, we have derived and proved the new sufficient conditions for the existence of robust optimal control and new explicit formulae (6) and (7), with the use of the *pseudo-inverse* matrices $D^+(t)$, that overcome computational difficulties inherent in not-completely controllable systems. The geometrical meaning of the appearance of the pseudo-inverse matrix $D^+(t)$ is that the trajectories $x(t)$ of the considered controlled system belong (exactly or approximately) to the subspace of dimensionality less than n.

References:

[1] M. Green, D.J. Limebeer, *Robust Linear Control.* Prentice Hall: NY, 1995.

[2] M.I. Grimble, M.A. Johnson, *Optimal Control and Stochastic Estimations: Theory and Applications, vols. 1, 2.* Wiley: New York, 1988.

[3] H.L. Trentelman, A.A Stoorvogel, M. Hautus, *Control Theory for Linear Systems.* Springer: London, 2001.

[4] T. Basar, P. Bernhard, H^∞*-Optimal Control and Related Minimax Design Problems. A Dynamic Game Approach (2nd ed).* Birkhauser: Boston, 1995.

[5] E.M. Khazen, Searching for Optimal Trajectory with Learning. *IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans* 2001; 31: 767 -774.

[6] E.M. Khazen, Reduction of dimensionality in choosing optimal control. *Int. J. Adapt. Control Signal Process.* 2003; 17: 1- 18.

[7] E. M. Khazen, Reduction of Dimensionality in Choosing Robust Optimal Control in Linear Systems. *WSEAS Transactions on Mathematics.* 2003; vol. 2, issue 4, 318 - 323.

[8] A. Bjorck, *Numerical Methods for Least Squares Problems.* SIAM: Philadelphia, PA, USA, 1996.