Numerical Solution of Inverse Eigenvalues Problem and its Application to Reconstruction of Illumination Optical Field

ALEXANDRE GREBENNIKOV, ANDREY S. OSTROVSKY
Facultad de Ciencias Físico Matemáticas
Benemérita Universidad Autónoma de Puebla
Av. San Claudio y Río verde, Ciudad Universitaria, CP 72570, Puebla
MÉXICO

Abstract:—The statements for two inverse eigenvalues problems for the compact positive self-adjoined operator are given. A numerical methods of its solution, based on the Descriptive Regularization, is proposed. A concrete realization of the proposed general scheme is realized as algorithm for reconstruction of the eigenvalues in the coherent-mode structure of the illumination field, using measurements of its radiometric characteristics. The algorithm is realized as software for MatLab system and illustrated with numerical experiments on simulated examples.

Key-Words:—Inverse problems, Regularization algorithms, Optical field

1 Formulation of the problems

Let A be the compact positive self-adjoined operator acting in the Hilbert space H. The direct spectral problem for this operator consists in constructing eigenvalues λ_n and eigenelements φ_n that satisfy the following equations:

$$A\varphi_n = \lambda_n \varphi_n, \quad n = 1,2,.... \tag{1}$$

It is well known [1] that solution of this problem exists and is presented as enumerable set of the orthogonal elements φ_n and positive eigenvalues $\lambda_n : 0 < ... < \lambda_{n+1} < \lambda_n < .... < \lambda_1$ converging to zero.

Let $\Lambda = \{\lambda_n, \varphi_n\}$ means the complete spectral structure set. The set Λ uniquely determines the operator A, because for any element u its image Au can be obtain due to the decomposition

$$Au = \sum_{n=0}^{\infty} \lambda_n (u, \varphi_n) \varphi_n \quad . \tag{2}$$

We suppose that, in general, the operator A is unknown. Let us consider the next statements of the inverse eigenvalues problems.

Problem 1 (Inverse eigenvalues problem). Let the eigenelements φ_n are known and the values of functionals $l_i(\Lambda) = f_i$, $i = 1,..., m$ are given. It is necessary to reconstruct the eigenvalues λ_n, n=1, 2, …

Sometimes the eigenelements φ_n for the operator A are unknown, but we know the eigenelements ψ_n for the "main part" of this operator. This situation appears also, if the real operator A is an approximation of some exact operator, the eigenelements for which are known. For such cases we introduce the "alternative spectral set" $\widetilde{\Lambda} = \{\mu_n, \psi_n\}$, where elements ψ_n are orthonormal, and we hope that $A\psi_n \approx \mu_n \psi_n, \quad n = 1,2,....$.

Problem 2 (Inverse alternative eigenvalues problem). Let the alternative eigenelements ψ_n are known and values of functionals $l_i(\widetilde{\Lambda}) = f_i$, $i = 1,..., m$ are given. It is necessary to reconstruct the "alternative eigenvalues" μ_n, n=1, 2, …

Stated problems are ill-posed [2], and their solutions must be considered as generalized solutions in the frame of some regularization scheme, which uses additional information about the set of desired incognitos values. We will use such additional information as belonging of eigenvalues $\{\lambda_n\}$ to some compact set K in Euclidian Space of infinite dimension.

2 Numerical Regularization Method

We will suppose bellow that functionals $l_i(\Lambda)$, $i = 1,...,\ m$ have a form

$$l_i(\Lambda) = \sum_{n=1}^{\infty} \lambda_n L_i(B\varphi_n), \quad i = 1,2,....,m, \qquad (3)$$

where L_i, *i=1,2,...* are lineal functionals, *B* is a given operator acting from *H* to *H.*

We propose the following numerical method for solution of Problem 1: to find among all real numbers $\{a_1,...,a_M\} \in K$ such M numbers $\overline{\lambda}_n \in K, n = 1,..., M$ that minimize the mean-square functional

$$\min_{\{a_1,...,a_M\}\in K} \sum_{i=1}^{m} | f_i - \sum_{n=1}^{M} a_n L_i(B\varphi_n) |^2 . \qquad (4)$$

For solution of Problem 2 the proposing numerical method is: to find among all real numbers $\{a_n\} \in K$ such

M numbers $\mu_n \in K, n = 1,..., M$ that minimize the mean-square functional

$$\min_{\{a_1,...,a_M\}\in K} \sum_{i=1}^{m} | f_i - \sum_{n=1}^{M} a_n L_i(B\psi_n) |^2 . \qquad (5)$$

It is well known that under made assumptions the problem of quadratic programming (4) (so as (5)) has the unique solution.

The presence of convention in minimization ascribe these methods to the class of Descriptive Regularization methods [3]. There are some effective algorithms to solve such problems [3]. But justification of the approximation quality of proposed methods in each concrete case (for the concrete operator *B* and functionals L_i) is rather difficult and requires a special investigation. We will apply bellow the proposed methods for solution of one optical problem and demonstrate the approximation quality by numerical experiments.

3 Application to solution of one optical problem

In [4] it is posed the problem of reconstructing the coherent-mode structure of an optic field from measurements of radiant intensity

$$J(s) = \sum_{n=1}^{\infty} \lambda_n |\Phi_n(s)|^2 . \qquad (6)$$

It is important to note that the quantities in the left-hand sides of equation (6) may be measured in physical experiment. In terms introduced above, functionals L_i are values of the function *J* in the nodes of a red on variable *s*. Operator *B* for the radiant intensity J is $B(\varphi_n) = |F(\varphi_n)|^2$, where F is the Fourier transform. The compact set K may be chosen as a set of positive vectors with monotone decreasing components or known convexity of the components.

For the considered optical problem the aim of the reconstruction is to approximate the cross-spectral density function that is determined by the exact formula

$$W(x_1, x_2) = \sum_{n=1}^{\infty} \lambda_n \varphi_n(x_1) \varphi_n^*(x_2) , \qquad (7)$$

using formula

$$\overline{W}(x_1, x_2) = \sum_{n=1}^{M} \overline{\lambda}_n \varphi_n(x_1) \varphi_n^*(x_2) ,$$

if the eigenfunctions φ_n are known, or, on the contrary, on the base of the formula with alternative spectral set

$$\widetilde{W}(x_1, x_2) = \sum_{n=1}^{M} \mu_n \psi_n(x_1) \psi_n^*(x_2) . \qquad (8)$$

4 Numerical experiments

We constructed a numerical algorithm for solution of Problems 1 and 2. For the numerical experiments we choose the example with

$$W(x_1, x_2) = \exp\left(-\frac{x^2{}_1 + x^2{}_2}{4\sigma_I^2}\right) \exp\left(-\frac{(x_1 - x_2)^2}{2\sigma_\gamma^2}\right) \qquad (9)$$

and known eigenvalues λ_n and eigenfunctions φ_n [5]. Detailed formulas for λ_n, φ_n and construction of alternative eigenfunctions ψ_n are presented in [4].

In computer simulation we used the radiant intensity J, changing in (6) infinite series for finite sum of *N* first members and calculating its values in *m* points as $f_i, i = 1,..., m$. The set K we choose as a set of positive convex vectors.

We realised the constructed algorithms as a software in MatLab, using a function *quadratic* of this system for quadratic programming [6]. In Fig. 1 and Fig. 2 we present the examples of solution of Problem 1 for *N*=40*, M*=20*, m*=51. In Fig. 3 - the example of the solution of Problem 2.

These examples illustrate a good approximation properties of proposed algorithms.

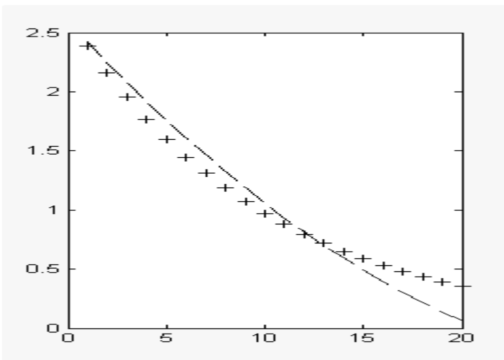

Fig. 1. Solution of Problem 1: + exact λ_n ;-- $\overline{\lambda}_n$.

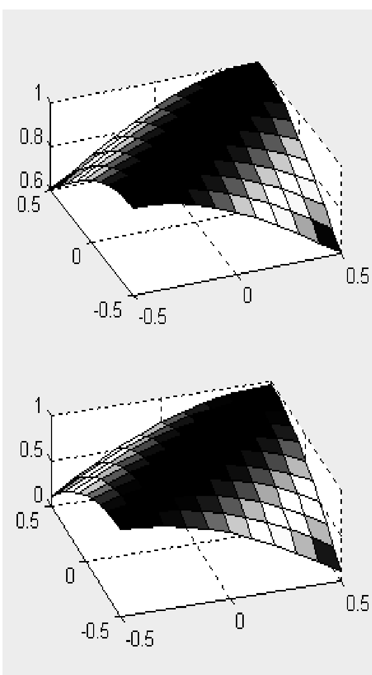

Fig. 2. Solution of Problem 1: above – exact W , bellow - reconstructed $\overline{W}$.

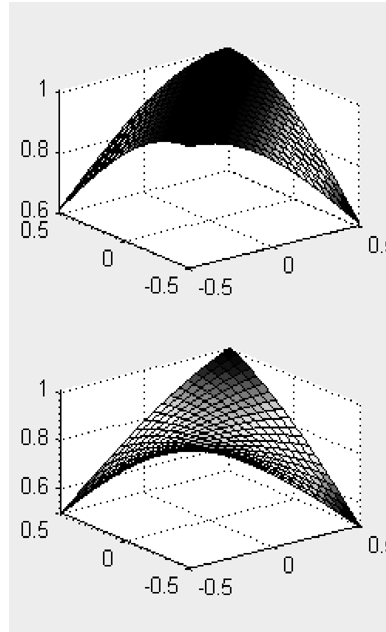

Fig. 3. Solution of Problem 2: above – exact W , bellow - reconstructed $\widetilde{W}$.

5 Conclusion

A numerical methods for solution of the inverse eigenvalues problem is proposed and applied for the reconstruction of optical field.

Authors acknowledge to the VIEP BUAP (Project No. II–105G04) and CONACYT of Mexico (Project No. 36875-E) for the support of this investigation.

Rreferences:

[1] N.I. Ahiezer and I.M. Glazman, *Theory of Linear Operators in the Hilbert Space,* Nauka, Moscow, 1966.

[2] A.N. Tikhonov at al., *Numerical Methods for the Soution of Ill-Posed Problems,* Kluwer Academic Publishers, Dordrecht, 1995.

[3] V.A. Morozov and A. I. Grebennikov, *Method for Solving Incorrectly Posed Problems: algorithmic Aspect,* Moscow State University Press, Moscow, 1992.

[4] A.S. Ostrovsky, A. I. Grebennikov and E. H. Garcia, Approximation of the Coherent-mode Structure of an Optical Field According to the Results of Physical Experiment. In present issue.

[5] L. Mandel and E. Wolf, Optical Coherence and Quantum Optics, Cambridge University Press, Cambridge, 1995.

[6] [http//:www.mathwork.com]