A Novel Approach for Solution of Boundary Problems for Differential Equations of Mathematical Physics

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Abstract— It is proposed a new approach for solution of the boundary problems for wide class of partial differential equations of mathematical physics, which includes the Laplace, Poisson and Helmholtz equations and parabolic equations. The approach is based on the discovered by author Local Ray Property and leads to new General Ray (*GR)* Method that uses the explicit analytical formulas with inverse Radon transformation. It is realized by fast algorithms and MATLAB software, which quality is demonstrated by numerical experiments.

Keywords -- Boundary problems for partial differential equations, Radon transformation.

1. Introduction

There are two main approaches for solving boundary problem for partial differential equations in analytical form: Fourier decomposition and the Green function method. The numerical algorithms are based on the Finite Differences method, Finite Elements (Finite Volume) method and the Boundary Integral Equation method. All methods and algorithms constructed on the bases of these approaches have some difficulties in realization for the complex geometrical form of the domain Ω . The Green function method is the explicit one [1], but for arbitrary coefficients of equations it is difficult to construct the Green function even for the simple geometry of Ω . Numerical approaches lead to solving systems of linear algebraic equations [2] that require a lot of computer time and memory. Hence, the development of new fast algorithms for solution of the considering problems is very actual.

We consider here a new approach on the base of the Local Ray Property (*LRP*), obtained by the author in [3], [4] for the stationary waves field. *LRP* leads to the explicit analytical formulas (*GR*-method) and fast algorithms, proposed firstly in [3] and founded numerically in [4] for the Dirichlet boundary problem with an arbitrary simply connected star domain Ω and continue contour Γ . Here this idea is applied to solution of more general equations and boundary conditions.

2. Boundary problems for elliptic equations

Here we consider the Dirichlet or Neumann boundary problems of solving the elliptic equation:

$$\left\{\begin{array}{l} \nabla(\varepsilon(x,y)\nabla u(x,y)) + k^2 u(x,y) = \psi(x,y), x, y \in \Omega; \\ \quad u(x,y) = f, \quad x, y \in \Gamma; \\ or \quad \dfrac{\partial u}{\partial n} = g \quad x, y \in \Gamma; \end{array}\right\} \quad (1)$$

with respect to the function $u(x,y)$ inside the plane domain Ω with a boundary Γ . Here $f(x,y)$ or $g(x,y)$ are given functions for $x, y \in \Gamma$, k is a real number, $\varepsilon(x,y) > 0$. If $k=0$, $\psi(x,y)=0$, we have the Laplace equation written in the divergent form. The case $\psi(x,y) \neq 0$ corresponds to Poisson equation. For $k \neq 0$ we have the Helmholtz equation.

The problem (1) describes the distribution of the "potential" function $u(x,y)$ for any field of stationary waves, which can be interpreted as electrostatic, elastic or optic field [1], [2].

3. Boundary problem for parabolic equations

Let us consider the boundary problem for the parabolic equation:

$$\left\{\begin{array}{l} u_t^{'}(t,x)=u_{xx}^{''}(t,x), \quad -1<x<1,\ t>0; \\ \quad u(0,x)=f_0(x), \quad -1\le x\le 1,; \\ u(t,-1)=f_{-1}(t), \quad u(t,1)=f_{+1}(t), \quad t\ge 0; \end{array}\right\} \qquad (2)$$

The specific element of this traditional statement is unbounded character of the domain Ω and a corresponding boundary Γ in range of variables *t* and *x*: $\Omega=\overline{\Omega}\equiv[0,\infty)\mathrm{x}[-1,1]$. We will suppose the finiteness of the solution in the domain $\overline{\Omega}$.

4. Local Ray Property (*LRP*)

Application of the *LRP* to the considering problems means to construct an analogue of equations (1), (2) describing the distribution of the function $u(x,y)$ and $u(t,x)$ along of general "Local Rays", which are presented by some straight line l with parameterization due a parameter τ : $x=p\cos\varphi-\tau\sin\varphi$, $y=p\sin\varphi+\tau\cos\varphi$, in the case of the elliptic equation, or with parameterization: $x=p\cos\varphi-\tau\sin\varphi$, $t=p\sin\varphi+\tau\cos\varphi$ in the case of the parabolic equation. Here $|p|$ is a length of the perpendicular, passed from the centre of coordinates to the line l, $\varphi\in[0,\pi]$ is the angle between the axis *x* and this perpendicular. Using this parameterization, we shall define functions $u(x,y)$ (and $u(t,x)$ for the equation (2)), $\varepsilon(x,y)$, $f(x,y)$, $g(x,y)$, $\psi(x,y)$ at $(x,y)\in l$ for fixed p, φ as functions $u(\tau)$, $\varepsilon(\tau)$, $f(\tau)$, $g(\tau)$, $\psi(\tau)$ of variable τ. We suppose that the domain Ω is a convex one and define for every fixed p and φ functions $u_0(p,\varphi)=u(\tau_0)$, $u_1(p,\varphi)=u(\tau_1)$, for parameters τ_0,τ_1, which correspond to the points of the intersection of the line l and boundary of the domain.

To formulate the Local Ray Property (*LRP*) let us consider the ordinary differential equation:

$$\begin{aligned} &\left(\varepsilon(\tau)u_\tau^{'}(\tau)\right)_\tau^{'}+k^2r^2(\varphi)u(\tau)=r^2(\varphi)\psi(\tau), \\ &\tau\in[\tau_0,\tau_1]; \quad r^2(\varphi)=\sin^2\varphi\cos^2\varphi, \end{aligned} \qquad (3)$$

as the local analogy of the equation (1), and corresponding analogy of the equation (2):

$$\begin{aligned} &u_{\tau\tau}^{''}(\tau)-K\,u_\tau^{'}(\tau)=0, \tau\in[\tau_0,\tau_1], \\ &K=\frac{\sin^2\varphi}{\cos\varphi}, \varphi\ne\frac{\pi}{2} \quad . \end{aligned} \qquad (4)$$

Boundary conditions lead to the corresponding local boundary conditions for $u(\tau)$ in points τ_0,τ_1. We will designate the solution of the local problem (3) or (4) with such boundary conditions as $\overline{u}(\tau)$. For standard domains as a circle, rectangular or $\overline{\Omega}$ it is simple to calculate τ_0,τ_1 so as functions $u_0(p,\varphi)$, $u_1(p,\varphi)$, on the boundary functions f, g, or f_0, f_{-1}, f_{+1}, and then obtain solution $\overline{u}(\tau)$ in explicit analytical form, using well known standard formulas for solution of ordinary differential equations.

Formulation of *LRP*:

the adequate local description of the solution u *on any straight line* l *(Local Ray) can be presented by the function* $\overline{u}(\tau)$*, so that the final global formula for the solution of equations (1) and (2) is true*

$$u(\xi,\eta)=R^{-1}\left[\int_{\tau_0}^{\tau_1}\overline{u}(\tau)d\tau\right],(\xi,\eta)\in\Omega\ , \qquad (5)$$

where R^{-1} *is inverse Radon transform,* $(\xi,\eta)=(x,y)$ *for the solution of elliptic equation (1),* $(\xi,\eta)=(t,x)$ *for the solution of parabolic equation (2)***.**

Formula (5) gives explicit solution for a considered class of boundary problems for arbitrary simply connected star domain Ω and continue contour Γ. We will concretise bellow formula (5) for particular cases of equations and demonstrate its validity by numerical examples.

5. Case of the Laplace equation with variable coefficients

For k=0 , $\psi(x,y)$=0, we obtain the mentioned analogue of equation (1) on the line l for every fixed p and φ the next ordinary differential equation

$$\left(\overline{\varepsilon}(\tau)\,\overline{u}_\tau^{'}(\tau)\right)_\tau^{'}=0\ . \qquad (6)$$

We introduce functions

$$\begin{aligned} &\gamma(\tau)=1/\overline{\varepsilon}(\tau); \quad k_0(\tau)=\int_{\tau_0}^{\tau_1}\gamma(\xi)d\xi; \\ &K_0(p,\varphi)=\int_{\tau_0}^{\tau_1}k_0(\xi)d\xi; \quad K_1(p,\varphi)=k_0(\tau_1). \end{aligned} \qquad (7)$$

Then, integrating twice equation (6), we obtain for solution of the Dirichlet problem the next formula

$$u(x,y) = R^{-1}[u_0(p,\varphi)(\tau_1 - \tau_0) + \frac{u_1(p,\varphi) - u_0(p,\varphi)}{K_1(p,\varphi)} K_0(p,\varphi)]. \quad (8)$$

Formula (8) presents the explicit *GR*-method for considering case. For the case $\varepsilon(x,y)=1$ we have more simple formula

$$u(x,y) = R^{-1}\left[\frac{u_1(p,\varphi)+u_0(p,\varphi)}{2}(\tau_1 - \tau_0)\right] . \quad (9)$$

Proposed method does not require solving any equations, because the Radon transform can be inversed by explicit fast formulas. Thus, constructed by author algorithm and corresponding computer software, which realize *GR*–method, are sufficiently fast, which is justified numerically in [4] for the case of Dirichlet boundary problem for Laplace equation with constant coefficients.

6. Reduction the case of arbitrary simply connected star domain to the unit circle

In [4] it was proposed a reduction of the Dirichlet problem for the Laplace equation for an arbitrary simply connected star domain Ω and continue contour Γ to the same problem on the unit circle. We make some change of variables, using equation for the curve Γ, which leads to the same problem with the standard Γ as the unit circumference. The mentioned transformation of coordinates does not require solution of any equations, does not include any bulky manipulation with complex variables. Hence, this transformation is realised by very fast algorithm, which is justified in [4] by numerical experiments for the sufficiently complex functions and domains.

We have generalized also the developed approach for the considering here class of boundary problems for elliptic equations (1). We can reduce such problems to the similar ones on the unit circle, using corresponding modifications of the coefficient k , functions $g(x,y)$, $\psi(x,y)$. Therefore we will present bellow the formulas and numerical examples of solution of the considering elliptic problems for the case of the unit circle. It is sufficiently to choose parameters τ_0, τ_1 for the unit circumference by formulas $\tau_{0,1} = \mp(1-p^2)^{1/2}$, then calculate functions $u_i(p,\varphi) = f(x^i, y^i)$, $x^i = p\cos\varphi - \tau_i \sin\varphi$; $y^i = p\sin\varphi + \tau_i \cos\varphi$; $i = 0,1$.

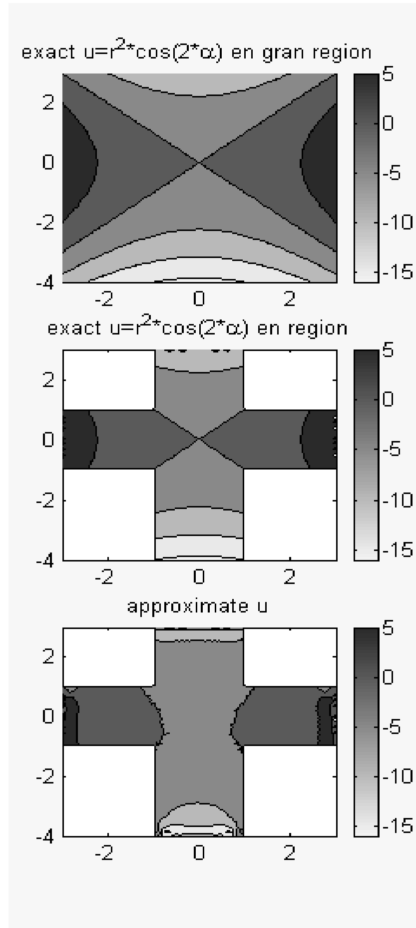

Fig. 1.

Here we present at Fig. 1 one new result for solution of the Dirichlet boundary problem for Laplace equation on the standard region as “cross”.

7. Case of Neumann boundary problem for Laplace equation

For the solution of the Neumann boundary problem we present here the formula, corresponding to the domain as unit circle and $\varepsilon(x,y)=1$:

$$u(x,y) = R^{-1}\left[(g_1(p,\varphi)+g_0(p,\varphi))(1-p^2)^{1/2}\right] + c, \quad (10)$$

where c is arbitrary constant, functions $g_i(p,\varphi)$, $i = 0,1$, correspond to the Neumann boundary condition function $g(x,y)$ in (1), calculated in the boundary points τ_0, τ_1. The unique solution can be obtained from function (10) by additional interpolation condition in one point on the boundary.

8. Case of the Poisson equation

The Dirichlet boundary problem for the Poisson equation corresponds to $k=0$, $\psi(x, y) \neq 0$. The main formula for its solution is the next one:

$$u(x,y) = R^{-1}\left[\frac{u_1(p,\varphi)+u_0(p,\varphi)}{2}(t_1-t_0)\right]+ \\ R^{-1}\left[r^2(\varphi)\left\{\psi_3(t_1)-\psi_3(t_0)-\frac{\psi_2(t_0)+\psi_2(t_1)}{2}(t_1-t_0)\right\}\right] \quad (11)$$

where $\psi_2(t)$, $\psi_3(t)$ are the second and the third primitive functions of the $\psi(t)$.

9. Case of the Helmholtz equation

For $k \neq 0$ we have the Helmholtz equation. We put for simplicity of the explication $\varepsilon(x, y) = 1$. For the no resonance case, when

$\bar{k} \equiv k\, r(\varphi)\sqrt{(1-p^2)} \neq \pi(1+2m)$, $m = 0, \pm 1, \pm 2, \ldots$,

the solution of the Dirichlet problem is given by formula:

$$u(x, y) = R^{-1}\left[\frac{u_1(p,\varphi)+u_0(p,\varphi)}{kr(\varphi)}tg(\bar{k}/2)\right]. \qquad (12)$$

The resonance case is just under the author investigation.

10. Case of the Parabolic Equation

The main formula for the solution is the next one:

$$u(t,x) = R^{-1}\left[\frac{u_1(p,\varphi)-u_0(p,\varphi)}{K}+u_0(p,\varphi)(\tau_1-\tau_0)\right]- \\ R^{-1}\left[\frac{u_1(p,\varphi)-u_0(p,\varphi)}{e^{K(\tau_1-\tau_0)}-1}(\tau_1-\tau_0)\right] \quad (13)$$

11. Results of numerical experiments

We have constructed the algorithmic and program realization of *GR*–method for considering types of problems in MATLAB system. We used the uniform discretization of variables $p \in [-1, 1]$, $\varphi \in [0, \pi]$, so as variables *x, y*, with *n* nodes. To calculate the inverse Radon transform for discrete data we constructed the original modification of *iradon* program from MATLAB package. We made testes on mathematically simulated model examples with known exact functions $u(x, y)$, $\varepsilon(x, y)$, $f(x, y)$, $g(x, y)$, $\psi(x, y)$.

Let us define as $u_n(x, y)$ the approximations obtained by formulas (8) - (13) for discreet case. We introduce the discrete relative medium estimations *rm* to demonstrate the quality of approximation:

$$rm(n) = \frac{1}{n^2}\frac{\sum_{i,j=1}^{n} |u_n(x_i, y_j) - u(x_i, y_j)|}{\max_{x_i, y_j \in \Omega} |u(x_i, y_j)|}.$$

Some results for solution the Dirichlet boundary problem for the Laplace equation on the unite circle for $\varepsilon(x, y) = 1/\cos(x+y)$, *n* = 101 are presented in Fig. 2.

One model example of solution by proposed algorithm of the Neumann problem for the Laplace equation is presented in Fig. 3. Numerical results for the Dirichlet problem for Poisson equation are presented in Fig. 4, 5.

For the Helmholtz equation one result of the numerical solution of model examples for *k*= 0.1 $\sqrt{2}$ is presented in Fig. 6.

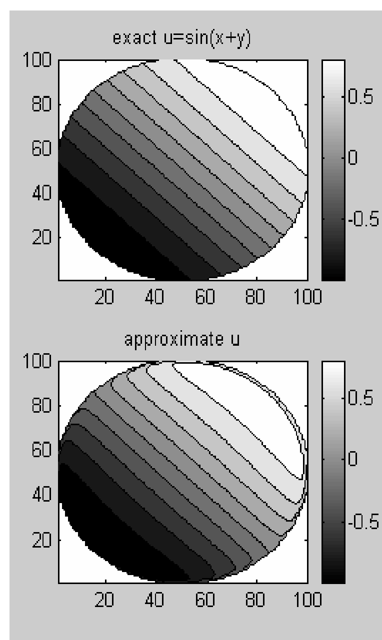

Fig. 2. *n*=101; $\varepsilon(x, y) = 1/\cos(x+y)$;
rm = 0.0452; reconstruction time t= 2.6870 sec.

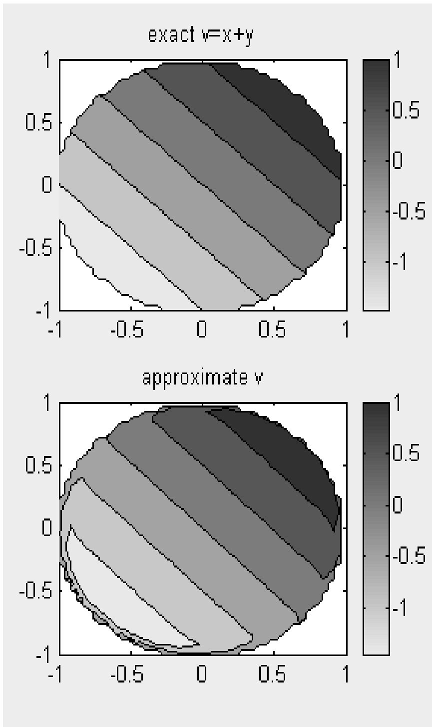

Fig. 3. Solution of the Neumann problem.

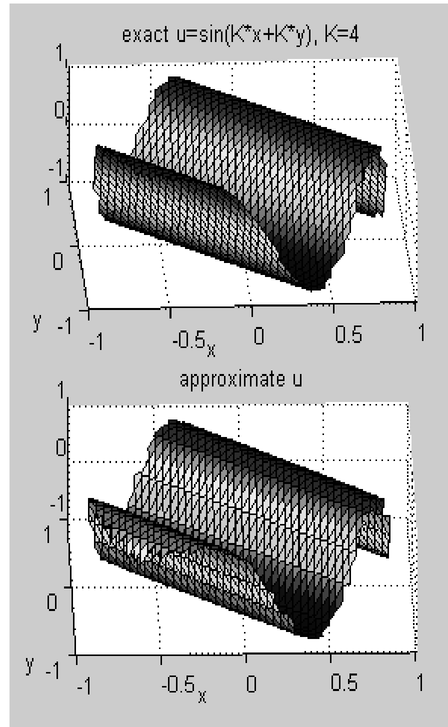

Fig. 5. n=50, rm = 0.1403,
reconstruction time t= 0.750.

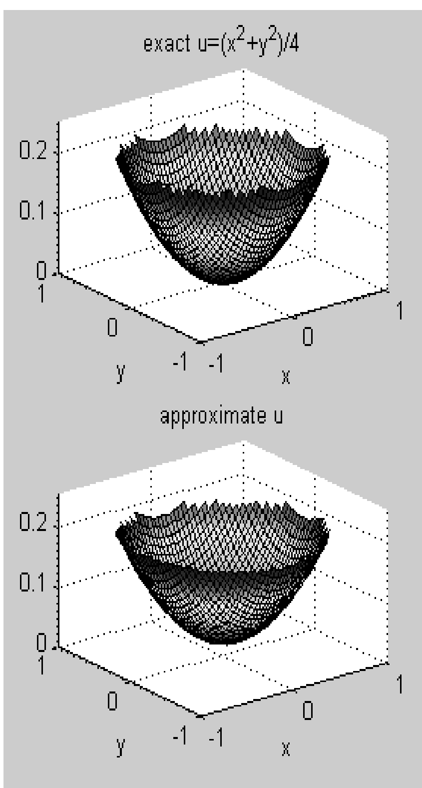

Fig. 4, n=50; rm = 0.0265;
reconstruction time t= 0.8120 sec.

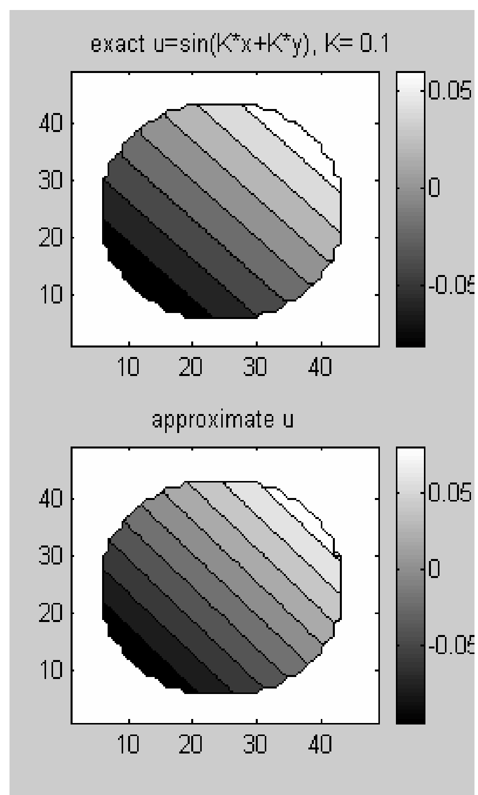

Fig. 6. Solution of the Helmholtz equation.

Some examples of solution of the parabolic equations are presented in Fig. 7 – 9.

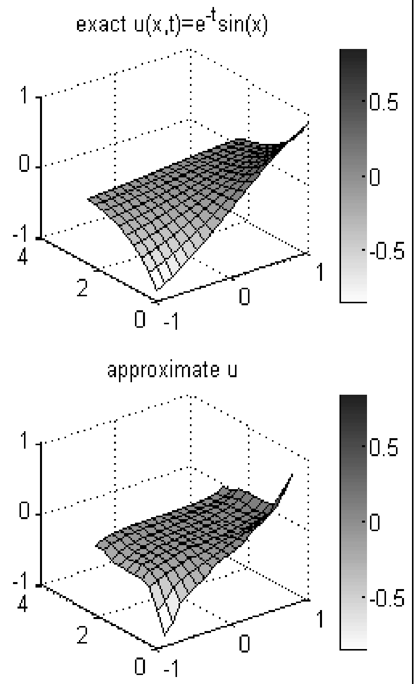

Fig. 7.

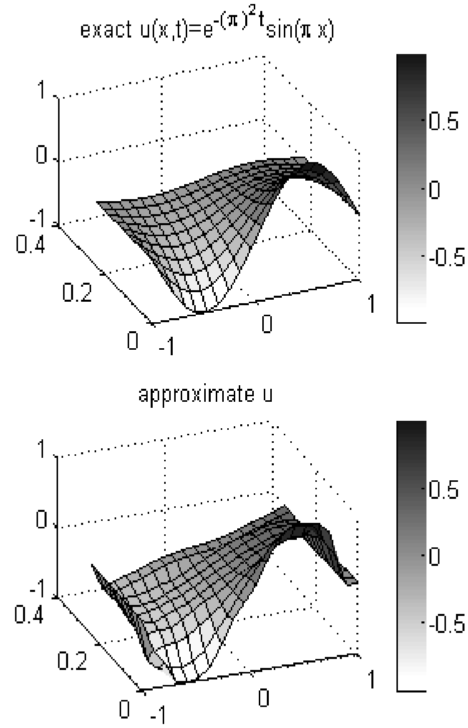

Fig. 9.

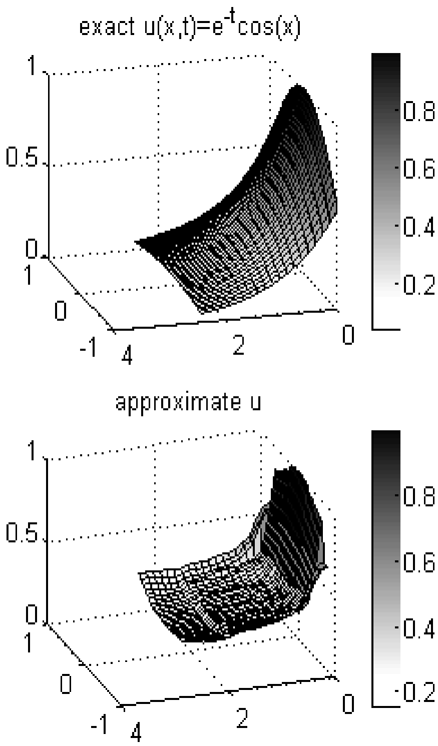

Fig. 8.

12. Conclusion and acknowledge

The new approach and *GR*-method for the solution of the boundary Dirichlet and Neumann problems for the elliptic and parabolic differential equations is proposed. The approximation and rapidity property of constructed algorithms are justified by numerical experiments. Developed approach can be applied to boundary problems for more general, including multidimensional, equations. Author acknowledges to VIEP BUAP for the partial support of the investigation.

Rreferences:

[1] S.L. Sobolev, *Equations of Mathematical Physics.*, Moscow, 1966.

[2] A.A. Samarsky, *Theory of Difference Schemes*, Moscow, 1977.

[3] A. Grebennikov. Fast Algorithm for Solution of Dirichlet Problem for Laplace Equation. ***WSEAS TRANSACTION on COMPUTERS Journal,*** Issue 4, Vol. 2, pp. 1039 -1043 (2003).

[4] A. Grebennikov, The Study of the Approximation Quality of *GR*-Method for Solution of Dirichlet Problem for Laplace Equation. ***WSEAS TRANSACTION on MATHEMATICS Journal,*** Issue 4, Vol. 2, pp. 312 -317 (2003).