An analysis of robustness on the synchronization of chaotic systems under nonvanishing perturbations using sliding modes*

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Abstract: - In this paper, an analysis for chaos synchronization under nonvanishing perturbations is presented. In particular, we use sliding modes control to synchronize perturbed chaotic systems. We use two coupled Rössler systems, the first like a master and the other like a perturbed slave. The proposed controller is able to synchronize perturbed chaotic systems, even with elimination of chattering problem. Some simulations are presented.

Key-Words: Chaotic synchronization, sliding modes control, Rössler system, nonvanishing perturbations.

1 Introduction

Synchronization of chaotic systems has been in the last years subject of great interest. Since the work of Pecora and Carroll [1], many researchers have proposed different approaches for chaotic synchronization [2]-[7]. Such interest is because chaotic synchronization is useful in many cases of practical interest, like to design secure communication systems [8]-[10], and significantly when it occurs in living systems like could be the case for the synchronization of the activity of groups of neurons located in different brain areas [11]-[13] or, in the synchronization between heart and respiratory rates [14] or, the coupling of biological oscillators [15].

In all these cases, it is important to be sure that the mechanisms that guide this synchrony are robust. For that reason, an important problem in the analysis of chaos synchronization is the robustness with respect to synchronization error, aging of physical components, uncertainties, and disturbances that exist in any realistic problem. The perturbation effect at the equilibrium point of the synchronization error dynamical system can be null or not. The first case, is called *vanishing perturbation*; this means that the equilibrium point of the perturbed synchronization error dynamical system remains the same as that of the equilibrium of the unperturbed one. The second case, is called *nonvanishing perturbation*, for which equilibrium points of the perturbed and unperturbed error systems are not the same. However, the perturbed error system may not have an equilibrium at all, in which case we cannot study the problem as the stability of equilibria any longer. So, the best we can expect are ultimately bounded state trajectories if the perturbation satisfies some conditions.

In this paper, we discuss the stability of the synchronization error between two coupled chaotic systems using sliding modes control from nonlinear control theory [16], [17] and subjected to a class of nonvanishing perturbations (see [18] and [19]), when the unperturbed error system has an uniformly asymptotically stable equilibrium point. We show that the error trajectories stay bounded if the perturbation satisfies some conditions. We use a classical example with a chaotic Rössler system and we show that, in this case, synchronizing partial states (output synchronization) of the Rössler system will result in the synchronization of their entire states.

This paper is organized as follows. In Section 2, we present the problem statement. In Section 3, we make a description of the sliding mode control for a nominal case (without perturbation), and we make an analysis of the perturbed case. In Section 4, we apply this result to synchronize a Rössler system for nominal and perturbed case using sliding modes control and making a quantitative analysis of the perturbation; some numerical simulations are presented. Finally, in Section 5, we give some concluding remarks.

2 Problem Statement

Let the dynamical master system be given by

$$\begin{aligned} \dot{x}_d &= f(x_d), \\ y_d &= h(x_d), \end{aligned} \tag{1}$$

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where $x_d(t) \in \mathbb{R}^n$, is the system state vector of the master system, $f(x_d)$ is a smooth vector field, and $y_d(t) \in \mathbb{R}$ is the output of the system.

Let us now take a dynamical system of the same order as that of (1),

$$\begin{aligned} \dot{x} &= f(x) + \Delta f(x) + Bu, \\ y &= h(x), \end{aligned} \tag{2}$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector of the slave system, B is a matrix of suitable size that define the control channel, $u(t) \in \mathbb{R}$ is the control input, and $\Delta f(x)$ is a perturbation term due to parameter mismatching or structure differences. The system (2), represents the perturbed case of the nominal slave system

$$\begin{aligned} \dot{x} &= f(x) + Bu, \\ y &= h(x). \end{aligned} \tag{3}$$

Assume that dynamical systems (1)-(3) under certain conditions have chaotic behavior. Then, the nominal slave chaotic system (3) *synchronizes* with the master chaotic system (1), if

$$\lim_{t\to\infty} \|x(t) - x_d(t)\| = 0, \tag{4}$$

no matter which initial conditions $x(0)$ and $x_d(0)$ have, and for suitable input signal $u(t)$.

The perturbed system (2) does not holds the condition (4), in which we cannot expect complete synchronization between systems (1) and (2), but we can expect ultimately bounded state trajectories, i.e.,

$$\lim_{t\to\infty} \|x(t) - x_d(t)\| \leq \rho, \tag{5}$$

with a small $\rho > 0$.

In the next sections, we describe how to satisfy synchronization conditions (4) and (5) from the perspective of the sliding modes control and Lyapunov-based stability from nonlinear control theory.

3 Description of the Sliding Mode Control

In order to obtain the objective of chaotic synchronization, from the control theory viewpoint, the synchronization problem can be seen as follows: define $e_i = x_i - x_{di}$. Then, the following system describes the dynamics of the unperturbed *synchronization error* between (1) and (3)

$$\begin{aligned} \dot{e} &= f(x) - f(x_d) + Bu, \\ y &= Ce. \end{aligned} \tag{6}$$

In this way, the synchronization problem can be seen as the stabilization of system (6) at the equilibrium point, i.e., the problem become to find a feedback control law $u(t)$ such that $e(t) \to 0$ (which implies that $x(t) \to x_d(t)$ and condition (4) holds) as $t \to \infty$. So, the proposed design method for the sliding mode control that stabilize (6) is as follows.

3.1 Sliding surface design and associated control law

Let us to propose the following sliding surface

$$s = e + \tilde{\lambda} \int_0^t e(\hat{r})\, d\hat{r}, \tag{7}$$

where $\tilde{\lambda} > 0$ it is a design constant that can be chosen suitably. Let us to find a continuous control such that under the initial position of the state $s(x, x_d)$, it yields identical equality to zero of the time derivative of vector $s(x, x_d)$ along trajectories of system (6)

$$\dot{s} = f(x) - f(x_d) + Bu + \tilde{\lambda} e = 0. \tag{8}$$

Suppose that a solution of the system of algebraic Eqs. (8) with respect to m-dimensional control does exist. This solution is referred as *equivalent control* $u_{eq}(x, x_d)$ [16], which assuming that matrix B is nonsingular for all x, it can be find an equivalent control from (8)

$$u_{eq} = -B^{-1} f(x) + B^{-1} f(x_d) - \tilde{\lambda} B^{-1} e. \tag{9}$$

Let the control law be represented as

$$u = u_{eq} + u_{sw}, \tag{10}$$

where $u_{sw} = -k\,\mathrm{sign}(s)$ is the switching control and the switching gain $k > 0$.

In order to find stability conditions and guarantee the existence of a sliding surface and the convergence at finite time of the trajectories of system (6) to the sliding surface (7), let the Lyapunov function of the system be [17]

$$V(s) = \frac{1}{2} s^2, \tag{11}$$

then, its first derivative with respect to time is

$$\dot{V}(s) = s\dot{s} = -k\|s\| \leq -\eta \|s\|, \tag{12}$$

with $k \geq \eta$ for some $\eta > 0$. Then $\dot{V}(s) \leq 0$, that is, there exist sliding mode dynamics and the trajectories of the system (6) converge to the surface in finite time. This is easy to demonstrate, since the solution to the Eq. (7), is given by

$$e = \hat{C} \exp(-\tilde{\lambda} t),$$

where $\hat{C}$ is a positive constant and the convergence time depends of both $\hat{C}$ and $\tilde{\lambda}$. Suppose that is wished a small synchronization error $e \leq \Delta_e$, then, the convergence time t_1 will be given by the equality

$$t_1 = \tilde{\lambda}^{-1} \ln \left(\hat{C} \Delta_e^{-1} \right). \tag{13}$$

Note: For chattering elimination it is proposed

$$u = \begin{cases} u_{eq} - k\, sign\,(s) & \text{for } \|s\| > \Delta_e \\ u_{eq} - \frac{\|s\|}{\Delta_e} & \text{for } \|s\| \leq \Delta_e \end{cases}. \tag{14}$$

Next, an analysis for the synchronization of the perturbed chaotic system (2) with (1) is presented.

3.2 Robust stability analysis of perturbed synchronization

Consider the perturbed chaotic system (2). It is desired that this one synchronize with the chaotic system (1). The following system describes the dynamics of the synchronization error in the perturbed case:

$$\begin{aligned} \dot{e} &= f(x) - f(x_d) + \Delta f(x) + Bu, \\ y_e &= Ce. \end{aligned} \tag{15}$$

If it is proposed the same surface (7) like from nominal case, then

$$\dot{s} = f(x) - f(x_d) + \Delta f(x) + Bu + \tilde{\lambda} e,$$

using again the control law (10), with the same equivalent control (9), we have

$$\dot{s} = -k\,\text{sign}\,(s) + \Delta f(x). \tag{16}$$

With the intention of demonstrate bounded trajectories in (15), for the proposed surface the following suppositions are established:

(H1) The origin of the sliding surface (7) is an asymptotically stable equilibrium point.

If supposition (*H1*) holds, then from an inverse Lyapunov's theorem the existence of a Lyapunov function $V(s)$ is guaranteed for (7). The function $V(s)$ satisfies

$$c_1 \|s\|^2 \leq V(s) \leq c_2 \|s\|^2, \tag{17}$$

$$\frac{\partial V}{\partial s}\frac{ds}{dt} \leq -c_3 \|s\|^2, \tag{18}$$

$$\left\| \frac{\partial V}{\partial s} \right\| \leq c_4 \|s\|, \tag{19}$$

for some positive constants c_1, c_2, c_3, and c_4 (for more details see [20]).

(H2) The perturbed term $\Delta f(x)$ satisfies the bound

$$\|\Delta f(x)\| \leq l_s + \delta_s \|s(x, x_d)\|. \tag{20}$$

for all $x, x_d, s \in \mathbb{R}^n$ and with positive constants l_s and δ_s.

Theorem 1 *Consider that the origin is an asymptotically stable equilibrium point for the nominal dynamical synchronization error system (6). Assume that supposition (H2) holds. Then, for all* $\|e(t_0)\| < \sqrt{\frac{c_1}{c_2}} r_B$*, there exist constants* $\bar{l}_s, \bar{\delta}_s > 0$ *such that the solution* $e(t)$ *of the perturbed error system (15) is ultimately bounded for all* $t > t_0$ *and for all perturbed term* $\Delta f(x)$ *that satisfies (20) with* $\delta_s < \bar{\delta}_s$*, and* $l_s < \bar{l}_s$.

Proof. Let be (11) candidate to Lyapunov function for the perturbed case and the sliding surface (7). Then, its first derivative with respect to time along perturbed error trajectories of system (15) is

$$\begin{aligned} \dot{V} &= s\left[-k\,\text{sign}\,(s) + \Delta f(x)\right], \\ &= -k\|s\| + s\Delta f(x), \end{aligned}$$

where $s\Delta f(x)$ it is the resulting term of the perturbation from system (2). Then, from (20), and using (17)-(19)

$$\begin{aligned} \dot{V} &\leq -c_3 \|s\|^2 + c_4 \|s\| (l_s + \delta_s \|s\|) \\ &\leq -(c_3 - c_4 \delta_s) \|s\|^2 + c_4 l_s \|s\|. \end{aligned}$$

If δ_s is small enough to satisfy the bound

$$\delta_s < \bar{\delta}_s \leq \frac{c_3}{c_4}, \tag{21}$$

then

$$\dot{V} \leq -a_s \|s\|^2 + c_4 l_s \|s\|,$$

with $a_s = c_3 - c_4 \delta_s > 0$. Moreover

$$\dot{V} \leq -(1 - \theta_s) a_s \|s\|^2 - \theta_s a_s \|s\|^2 + c_4 l_s \|s\|,$$

where $0 < \theta_s < 1$, and a bound for l_s is given by

$$l_s < \bar{l}_s \leq \frac{\theta_s a_s}{c_4} \|s\|. \tag{22}$$

Then,

$$\dot{V} \leq -(1 - \theta_s) a_s \|s\|^2, \qquad \forall \|s\| \geq \mu = \frac{c_4 l_s}{\theta_s (c_3 - c_4 \delta_s)},$$

and an ultimate bound is finally given by

$$b_s \leq \frac{c_4 l_s}{\theta_s (c_3 - c_4 \delta_s)} \sqrt{\frac{c_2}{c_1}}. \tag{23}$$

■

Note that the ultimate bound b_s is proportional to the upper bound on the perturbation l_s. This result can be viewed as a robustness property of the unperturbed synchronization error system having asymptotically stable equilibria at the origin because it shows that arbitrarily small nonvanishing perturbations will not result in large synchronization error.

4 Synchronization of Perturbed Chaotic Systems Using Sliding Modes Control

We make use of the previous result to show how the synchronization of chaotic systems can be achieved. We consider the output synchronization problem where the system (6) has a strong relative degree $r = 1$, i.e., $L_g h(e) \neq 0$ (for all e and with $g(e) = B$, and $h(e) = Ce$). Even though it yields a zero dynamics $\xi(0, \zeta)$ for the system (15), several chaotic systems are so-called minimum phase, that is, the zero dynamics converge to an attractor, the closed-loop system is internally stable [21]. This is reasonable for the boundness of chaotic attractor in state space and the interaction of all the trajectories inside the attractor. So, when we have taken actions to achieve $e_{i^*} \to 0$, for a suitable $i^* \in \{ i \mid r = 1 \}$, the part $\xi(e, \zeta) \to \xi(0, \zeta) \to 0$ asymptotically for the so-called minimum-phase character (see [22] for an illustrative example). So, we only need to synchronize one state and the others will be synchronized automatically (for unperturbed case).

Then, consider the Rössler system writing in the form (1) given by [23]:

$$\begin{pmatrix} \dot{x}_{d1} \\ \dot{x}_{d2} \\ \dot{x}_{d3} \end{pmatrix} = \begin{pmatrix} -x_{d2} - x_{d3} \\ x_{d1} + \hat{\alpha} x_{d2} \\ \hat{\alpha} + x_{d3}(x_{d1} - \hat{\mu}) \end{pmatrix}, \quad y_d = x_{d2}, \tag{24}$$

as a chaotic master system. With the parameter values $\hat{\alpha} = 0.2$ and $\hat{\mu} = 7$, the Rössler system exhibits chaotic dynamics. Consider as a perturbed slave chaotic system to another Rössler system in the same way

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -x_2 - x_3 \\ x_1 + \tilde{\alpha} x_2 \\ \tilde{\alpha} + x_3(x_1 - \tilde{\mu}) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u, \quad y = x_2, \tag{25}$$

where $\tilde{\alpha} = \hat{\alpha} + \Delta\hat{\alpha}$, and $\tilde{\mu} = \hat{\mu} + \Delta\hat{\mu}$, with $\Delta\hat{\alpha}$ and $\Delta\hat{\mu}$ like small perturbations in the parameters $\hat{\alpha}$ and $\hat{\mu}$, respectively. For this particular case, we have chosen $\Delta\hat{\alpha} = 0.02 \sin(2\pi t/T)$ and $\Delta\hat{\mu} = 0.7 \sin(2\pi t/T)$, with $T = 10$, such that $\tilde{\alpha} = 0.2 + 0.02 \sin(2\pi t/T)$ and $\tilde{\mu} = 7 + 0.7 \sin(2\pi t/T)$, represent the perturbed parameters with variations between $\pm 10\%$, for the slave chaotic system. So, the perturbed slave chaotic system can be writing in the form (2), as follows

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -x_2 - x_3 \\ x_1 + \hat{\alpha} x_2 \\ \hat{\alpha} + x_3(x_1 - \hat{\mu}) \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta\hat{\alpha} x_2 \\ \Delta\hat{\alpha} - \Delta\hat{\mu} x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u, \quad y = x_2. \tag{26}$$

Suppose that it is desired that the output $y(t) = x_2(t)$ of (26) follows the output trajectory $y_d(t) = x_{d2}(t)$ of (24). Considering that these small variations in the parameters or parameter mismatching are not available for measurement or are simply not considered in the design of the controller, then, we can design the controller from the unperturbed case, i.e., $\Delta\hat{\alpha} = \Delta\hat{\mu} = 0$. Then, consider the dynamics of the synchronization error

$$\begin{aligned} \dot{e}_1 &= -(e_2 + e_3) + u, \\ \dot{e}_2 &= e_1 + \hat{\alpha} e_2, \\ \dot{e}_3 &= x_3 e_1 + (x_1 - \hat{\mu}) e_3 - e_1 e_3, \\ y_e &= e_2. \end{aligned} \tag{27}$$

A sliding surface for (27) based on the output synchronization error $y_e(t) = e_2(t) = x_2(t) - x_{d2}(t)$, is proposed as

$$s = \dot{e}_2 + \tilde{\lambda} e_2. \tag{28}$$

Its first derivative with respect to time along (27) yields the following sliding mode equation

$$\dot{s} = \left(\hat{\alpha} + \tilde{\lambda}\right) e_1 - \left(1 - \hat{\alpha}\tilde{\lambda} - \hat{\alpha}^2\right) e_2 - e_3 + u, \tag{29}$$

where, in agreement with (10), and choosing $\tilde{\lambda} = \hat{\alpha}^{-1} - \hat{\alpha} > 0$ (since $\hat{\alpha} < 1$), the control law $u(t)$ is given by

$$u = \frac{1 - 2\hat{\alpha}^2}{\hat{\alpha}} e_1 + e_3 - k \operatorname{sign}\left(e_1 + \frac{1}{\hat{\alpha}} e_2\right). \tag{30}$$

Then, replacing the control law (30) in Eq. (29) we obtain $\dot{s} = k \operatorname{sign}(e_1 + \frac{1}{\hat{\alpha}} e_2)$. Thus, by (11) it is guaranteed that the surface (28) is sliding and the output of (27) converges to the sliding surface, which means that the output of slave chaotic system (26) converges to the output of the master chaotic system (24), when $\Delta\hat{\alpha} = \Delta\hat{\mu} = 0$.

This result is illustrated with some numerical simulations. The initial conditions $x(0)$ and $x_d(0)$ were $(1.1, -0.8, -0.8)$ and $(0.1, -0.3, 1)$, respectively, $k = 1$ and $\lambda = 4.8$. Fig. 1 shows a) the output of (26), $y(t) = x_2(t)$ following the output of (24), $y_d(t) = x_{d2}(t)$, b) the convergence of $e_1(t)$, $e_2(t)$, and $e_3(t)$ to sliding surface $s(t) = 0$, c) the sliding surface converging to zero, and d) the bounded control $u(t)$. The control law takes action intentionally after 60 seconds with the purpose of making more illustrative the individual dynamics and their behavior after the coupling. As it was discussed previously, synchronization in all states is obtained.

Once we have been able to design a control law that achieve master-slave synchronization we are ready to apply this control law in the perturbed case, i.e., that $\Delta\hat{\alpha} \neq \Delta\hat{\mu} \neq 0$, when they take the values established before, i.e., $\Delta\hat{\alpha} = 0.02 \sin(2\pi t/T)$ and $\Delta\hat{\mu} = 0.7 \sin(2\pi t/T)$. Then, using the same control law from the unperturbed case, and using (21) and (22), the closed-loop perturbed error

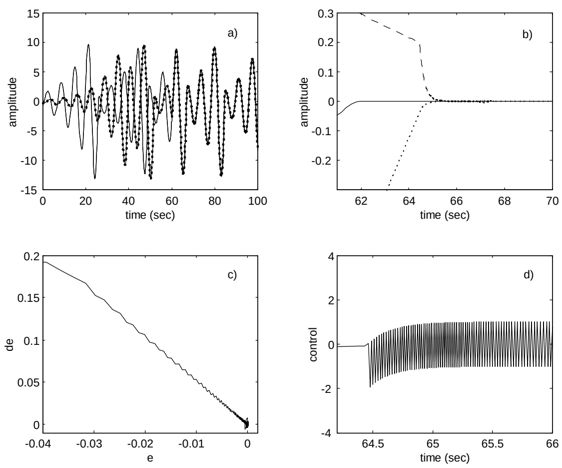

Figure 1: Synchronization of Rössler system: unperturbed case. a) The slave's output $y(t) = x_2(t)$ following the master's output $y_d(t) = x_{d2}(t)$. b) Convergence of $e_1(t)$, $e_2(t)$, and $e_3(t)$ to sliding surface $s(t) = 0$. c) Sliding surface. d) Bounded control $u(t)$. The control law takes action intentionally after 60 seconds.

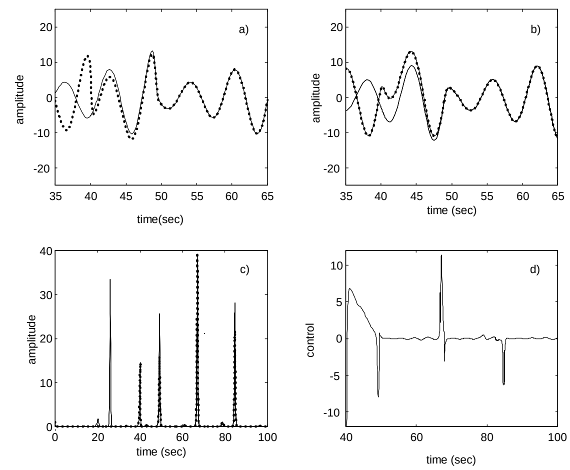

Figure 2: Synchronization of Rössler system: perturbed case. a) $x_1(t)$ following $x_{d1}(t)$. b) $x_2(t)$ following $x_{d2}(t)$. c) $x_3(t)$ following $x_{d3}(t)$. d) Control without chattering $u(t)$. Control takes action after 40 seconds.

system (15) leads to (29) taking the form (16), and the correspond Lyapunov function (11) satisfies (17)-(19) with $c_1 = 0.25$, $c_2 = 0.75$, $c_3 = k = 5$, and $c_4 = 1$. Moreover, it has be considered that the maximum value of the perturbation $\Delta_{\max}$ is $|\Delta|_{\max} = 0.7$. So, a bound $\delta_s = 4$ and the constant value $\theta_s = 0.2$ leads the bound $l_s = 0.44735$. Finally, in agreement with Theorem 1, if we take a ball of radio $r_B = 4$, an ultimate bound for $e(t)$ is given by (23) like $b_s = \|s\|_{\max} = 3.8742$.

Fig. 2 shows the limited behavior of the perturbed synchronization error $e_1(t)$, $e_2(t)$, and $e_3(t)$ for different initial conditions and the control law (14) without chattering taking action after 40 seconds. Fig. 3 shows a) the bounded synchronization error inside $b_s = 3.8742$ after a short transient time and b) the sliding surface without chattering after Δ_e.

5 Concluding Remarks

In this paper, a sliding mode control for synchronizing chaotic systems under nonvanishing perturbations is proposed. Based on a mathematical analysis and Lyapunov stability theory, a sliding mode controller is designed such that the slave chaotic system under perturbations can be synchronized with a master chaotic system like the desired chaotic trajectory, no matter which initial conditions they have. The Rössler system was used as an example to verify and visualize the synchronization strategy. Under the proposed control method, the synchronization error converge at a finite short time to the sliding surface. A control law with elimination of chattering has been proposed for the perturbed case and the synchronization error holds the ultimate calculated bound. Both analysis and numerical simulation reveal that the proposed sliding mode control has great potential for synchronizing chaotic systems under nonvanishing perturbations.

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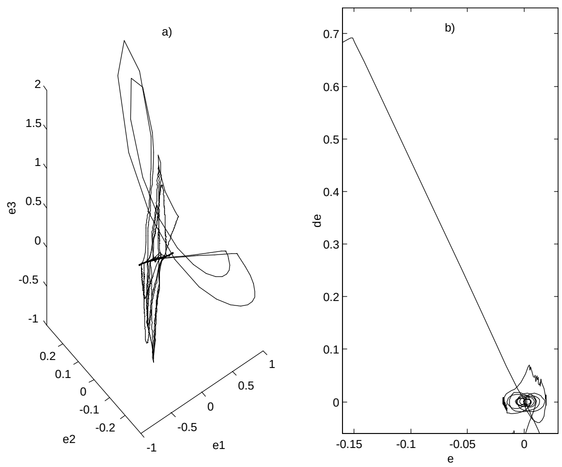

Figure 3: Synchronization of Rössler system: perturbed case. a) Bounded synchronization error inside $b = 3.8742$ after a short transient time. b) Sliding surface without chattering after Δ_e.

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