The evolution of a single species food-limited population model with delay: A numerical study

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Abstract: In this work we model and numerically analyse, the dynamics of a single species food-limited model with delay, we focus in the asymptotic behavior. We use the sub and supersolutions method to provide some numerical simulations of the asymptotic behavior for several cases, autonomous, periodic, almost periodic concluding with a model that incoporates nonlocal delays as well as a continuous delay.

Key-Words: Reaction-diffusion, asymptotic behaviour, continuous delay, nonlocal effect, population model.

1 Introduction.

Food limited population models have been around since the work of [1] and [2] back to the sixties. To have a more realistic model, several reserchers have used a reaction-diffusion equation in order to incorporate spatial dispersal and enviromental heterogeneity [3] and [4]. A detailed account can also be found in the recent work, [5]. In [6], [7], [8], [9] and [10], we have seen a delay effect incorporated in the reaction-diffusion model as well as non local delay effect.

We consider the equation

$$u_t - \alpha u_{xx} = F(x,t,u,f*u), \quad x \in [0,1], t \in \mathbb{R} \qquad (1)$$

for different families of functions F. We will deal with the cases of F being autonomous, periodic, quasiperiodic and $f*u$ depends on values on the past as well as on the non local variation.

We will be dealing with the unidimensional case $\Omega = [0,1]$. The function u is also subject to a Dirichlet condition

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t \in \mathbb{R} \qquad (2)$$

and the values for u in negative time are given as initial condition

$$u(x,s) = \eta(x,s), \qquad x \in \Omega,\ s \in \mathbb{R}^-,\ \eta(x,s) \text{ known.} \qquad (3)$$

The numerical solutions will be found using the method constructed in [11].

2 Problem Formulation.

2.1 Discrete delay.

The general form for the function F that models the discrete delay is

$$F(x,t,u,f*u) = r(x,t)u(x,t)\frac{K(x,t) - (au(x,t) + bu(x,t-\tau))}{K(x,t) + c(x,t)(au(x,t) + bu(x,t-\tau))} \qquad (4)$$

where a, b are non negative constants such that $a + b = 1$.

The parameter τ is the delay, which allows us to model dependency on the actual state as well as some state in the past. We will use $\tau = 1$. Varying the parameters a, b allows us to put more weight on the actual state versus the delayed. Also notice that if $a = 1, b = 0$ then the delay effect dissapears. We will treat three different cases

i. Autonomous case

ii. Periodic case

iii. Almost periodic case

2.2 Continuous delay.

The previous model can be improved in several ways. First, it can be noted that populations not only depend on the current state and some previous fixed delay, but depend on the previous states. So instead of taking a constant delay τ, we allow it to run from $-\infty$ to the present time t.

Also, species do not remain still on some location as time changes, but they move within their habitat. This introduces non locality in our model.

In order to take these two considerations into account, we use the model presented by Gourley in [10], for constant coefficients. The function F has now the form

$$F(x,t,u,f*u) = r(x,t)u(x,t)\frac{K(x,t) - (au(x,t) + b(f*u)(x,t))}{K(x,t) + c(x,t)(au(x,t) + b(f*u)(x,t))} \qquad (5)$$

where $f*u$ is the following convolution

$$(f*u)(x,t) = \int_{-\infty}^{t}\int_{\Omega} G(x,y,t-s)f(t-s)u(y,s)\,dy\,ds. \qquad (6)$$

Here f is such that $\int_0^\infty f(x) = 1$ and represents the corresponding weight for each of the past states into the model. Following [10], we use the strong generic kernel

$$f(t) = \frac{te^{-t/\tau}}{\tau^2}, \qquad (7)$$

and

$$G(x,y,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-\alpha n^2 t}\sin(nx)\sin(ny) \tag{8}$$

3 Problem Solution

We will numerically solve the equations (5)-(6) using the method of sub and supersolutions [11].

In order to do that, we first describe Pao's process [11]. We consider the following continuous parabolic boundary-value problem in a bounded domain Ω in $\mathbb{R}^p$ ($p = 1,2,\ldots$):

$$\begin{aligned} u_t - \alpha\nabla^2 u &= f(x,t,u), \quad x\in\Omega, 0<t\le T,\\ B[u(x_0,t)] &= h(x_0,t), \quad x\in\partial\Omega, 0<t\le T\\ u(x,0) &= \psi(x), \quad x\in\Omega. \end{aligned} \tag{9}$$

Then, by using the standard second order difference notation

$$\Delta^{(\nu)}u_{i,j} \equiv \frac{u(x_i+h_\nu e_\nu,t_j) - 2u(x_i,t_j) + u(x_1-h_\nu e_\nu,t_j)}{h_\nu^2} \tag{10}$$

the finite difference version of the continuous system (9) is given by[1]

$$L[u,0] = f_{i,j}(u_{i,j}), \quad (i,j)\in\Lambda_p, \tag{11}$$

$$B[u_{0,j}] = h_{0,j}, \qquad i=0\in S_p, j=1,\ldots,M, \tag{12}$$

$$u_{i,0} = \psi_i, \qquad i\in\Omega_p \tag{13}$$

where $h_\nu e_\nu \equiv (0,\ldots,0,h_\nu,0,\ldots,0)\in\mathbb{R}^p$ and Ω_p, Λ_p, S_p are the sets of all mesh points in $\Omega, \Omega\times(0,T]$ and $\partial\Omega$ respectively, and $\overline{\Lambda}_p$ is the set of all mesh points in $\overline{\Omega}\times[0,T]$.

We will construct a monotone sequence for the finite difference system (11)-(13) in a multidimensional domain. We first choose a suitable function $\gamma_{i,j}\equiv\gamma(x_i,t_j)\le 0$ and add the term $\gamma_{i,j}$ on both sides of (11). The choice of γ depends on the reaction function f. Then by starting from a suitable initial iteration $u_{i,0}^{(0)}$ we successively construct a sequence $\{u_{i,j}^{(m)}\}$ from the linear system

$$\begin{aligned} L[u^{(m)},\gamma] &= \gamma_{i,j}u_{i,j}^{(m-1)} + f_{i,j}(u_{i,j}^{(m-1)}), \quad (i,j)\in\Lambda_p\\ B[u_{0,j}^{(m)}] &= h_{0,j} \qquad i=0\in S_p, j=1,\ldots,M,\\ u_{i,0}^{(m)} &= \psi_i \qquad i\in S_p, \end{aligned} \tag{14}$$

for $m = 1,2,\ldots$. The main idea for this construction is to characterize a class of initial iterations so that the corresponding sequence obtained from (14) converges monotonically to a solution of the system (11)-(13).

Definition 1 *A function $\tilde{u}_{i,j}$ defined on Λ_p is called upper solution of (11)-(12) if it satisfies the inequalities*

$$\begin{aligned} L[u,0] &\ge f_{i,j}(\tilde{u}_{i,j}), \quad (i,j)\in\Lambda_p,\\ B[\tilde{u}_{0,j}] &\ge h_{0,j}, \qquad i=0\in S_p, j=1,\ldots,M,\\ \tilde{u}_{i,0} &\ge \psi_i, \qquad i\in\Omega_p \end{aligned} \tag{15}$$

[1] $L[u,\gamma] = k_j^{-1}(u_{i,j}-u_{i,j-1}) - \sum_{\nu=1}^{p} d_{i,j}\Delta^{(\nu)}u_{i,j} + \gamma_{i,j}u_{i,j}$

Similarly, $\hat{u}_{i,j}$ is called a lower solution of (11)-(12) if it satisfies all the reversed inequalities in (15).

It is clear from the above definition that every solution $u_{i,j}$ is an upper solution as well as a lower solution. Suppose upper and lower solutions exist and $\hat{u}_{i,j}\le\tilde{u}_{i,j}$ on $\overline{\Lambda}_p$. Then by using $u_{i,j}^{(0)} = \tilde{u}_{i,j}$ and $u_{i,j}^{(0)} = \hat{u}_{i,j}$ we can obtain two sequences from (15). Denote these two sequences by $\{\overline{u}_{i,j}^{(m)}\}$ and $\{\underline{u}_{i,j}^{(m)}\}$ respectively, and refer to them as maximal and minimal sequences. It can be shown that under some one-sided Lipschitz condition of f, the maximal and minimal sequences converge monotonically to a solution of (11)-(13).

The conditions imposed on f are given as follow:

There exist functions $\gamma_{i,mj}\ge 0$, $\sigma_{i,j}\ge 0$ on $\overline{\Lambda}_p$ such that

$$f_{i,j}(w) - f_{i,j}(w) \ge -\gamma_{i,j}(w-v) \quad \text{for } \hat{u}\le v\le w\le\tilde{u} \tag{16}$$

$$f_{i,j}(w) - f_{i,j}(w) \le -\sigma_{i,j}(w-v) \quad \text{for } \hat{u}\le v\le w\le\tilde{u}. \tag{17}$$

Notice that if f satisfies the Lipschitz condition

$$|f_{i,j}(w) - f_{i,j}(v)| \le k_{i,k}|w-v| \quad \text{for } \hat{u}\le v\le w\le\tilde{u}$$

then both (16) and (17) are satisfied with $\gamma_{i,j} = \sigma_{i,j} = k_{i,j}$. The following theorem is proved in [11]:

Theorem 1 *Let $\tilde{u}_{i,j}$ and $\hat{u}_{i,j}$ be a pair of upper and lower solutions such that $\hat{u}_{i,j}\le\tilde{u}_{i,j}$ on $\overline{\Lambda}_p$ and let f satisfy condition (16). Then the maximal sequence $\{\overline{u}_{i,j}^{(m)}\}$ converges monotonically from above to a solution $\overline{u}\equiv\overline{u}_{i,j}$ and the minimal sequence $\{\underline{u}_{i,j}^{(m)}\}$ converges monotonically from below to a solution $\underline{u}\equiv\underline{u}_{i,j}$ of (11)-(13). Moreover, $\overline{u}$ and $\underline{u}$ satisfy the relation*

$$\hat{u}\le\underline{u}^{(1)}\le\underline{u}^{(2)}\le\cdots\le\underline{u}\le\overline{u}\le\cdots\le\overline{u}^{(2)}\le\overline{u}^{(1)}\le\tilde{u}. \tag{18}$$

If, in addition, f satisfies condition (17) with $\sigma_{i,j}\le k_j^{-1}$ then $\overline{u} = \underline{u}$ and is the unique solution such that (18) holds.

3.1 Discrete delay.

For this case, in [8] was considered autonomous coefficients depending on the space variable and they prove that as long as $a > b$, that is, the weight of the population at present time is larger that the weight at the delayed time, then both ecuations, with delay and with no delay, have the same steady state. This behavior could be reproduced with several functions, among others

$$\begin{aligned} r(x,t) &= 6 + 3\sin(\pi x) \\ K(x,t) &= 10.3 - 5\sin(\pi x) \\ c(x,t) &= 5.3 - 2\cos(\pi x) \\ \eta(x,s) &= 0.2e^{0.1s}\sin(\pi x) \end{aligned} \tag{19}$$

In a similar way, [9] considered heterogeneous coefficients periodic in the time variable. They showed that for $a > b$, the cases with delay and with no delay have the same global attractor, this could be reproduced with

$$K(x,t) = 15 + 2\sin(2\pi t) \tag{20}$$

with r, c and η as in the autonomous case mentioned above.

Moreover, it could be noticed that as a decreases, the irregularities become more prominent and for $a < b$ we see the presence of an oscillating behaviour, more prominent as a becomes smaller. Population seems to suddenly increase by large amounts just to decrease soon near zero for some time, after a while the sudden increase-decrease occurs followed by some low period and so on. We found that the times at which the oscillations appear does not depend on the values for a, b but rather on the values of the delay. A smaller delay makes such waves to appear earlier, whereas a larger value will delay their presence.

When the current state weight becomes null, that is, giving the full weight to the delay by setting $a = 0$ and $b = 1$ the periodic behaviour given by F has dissapeared leaving only the oscillations caused by the delay (sudden increases followed by some very low ranges).

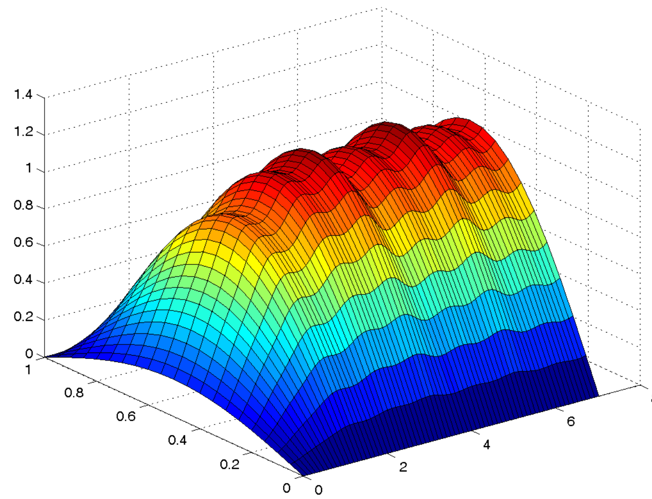
3.1.1 Almost periodic case.

The next step consists on taking almost periodic functions. For this case, no analytical results are known yet. These functions do not present a period but their behaviour is somehow regular. Examples of almost periodic functions are sums of periodic functions with noncommesurable periods. The simulations were done using

$$K(x,t) = 15 + 2\sin(2\pi t) + \sin(\sqrt{2}t) \tag{21}$$

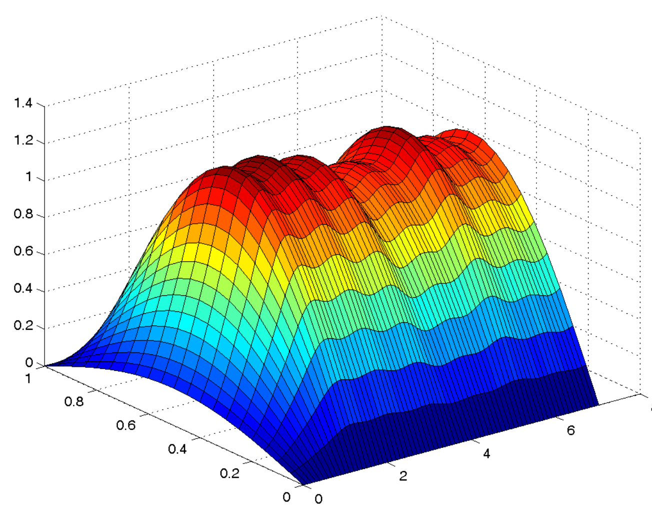
Starting with no delay, we see that the functions behaves similarly to the periodic case, but we see the effect of both periods in action.

The effect of putting more weight to the delay than the current state is similar to the effect seen in the previous cases. The case with no delay is illustrated in the following figure.



Parameter $a = 1.0$**, no delay.**

For large values of a, delay introduces a small disturbance but the current state dominates in the long term but as a decreases and b increases, the solutions become less regular and for even smaller values for a, the periods seen in the non delay solution dissapear and the strange oscillations appear again.



Parameter $a = 0.6$**, discrete delay.**

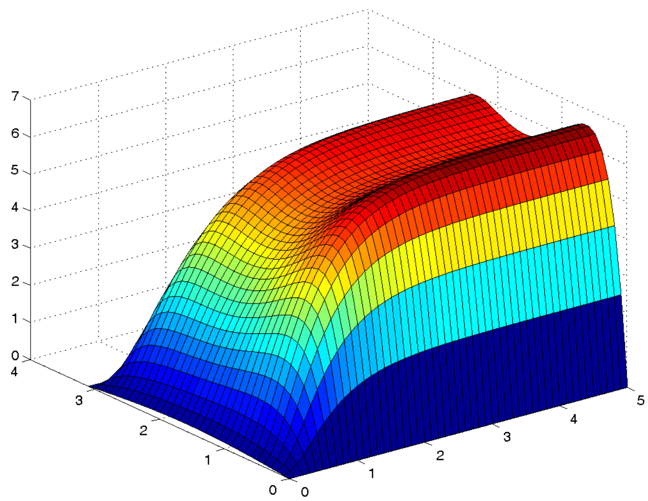
Changing the other functions, $r(x,t)$ and $c(x,t)$, we have similar effects to the ones shown here.

3.2 Continuous delay and non local effect.

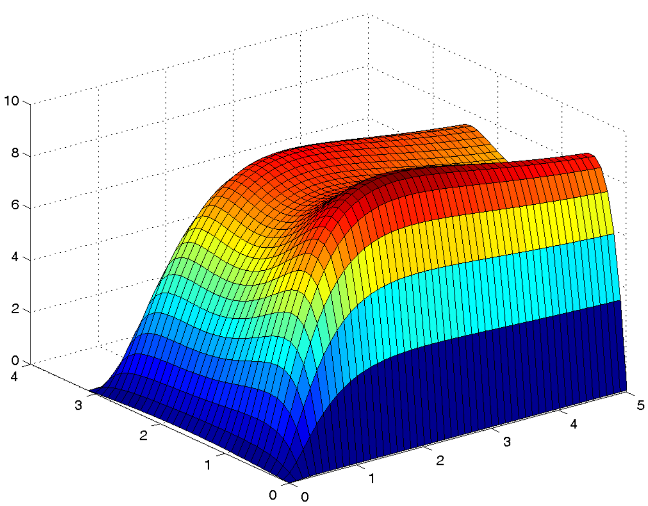
For this case, few results are known comparing the asymptotic behavior of the case with delay and the one without it like those presented in [10] in which they work with constant coefficients. For the following cases no analytical results are known yet.

3.2.1 Heterogenous coefficients

These two graphs, the first one with $a = 1$, that is, with no delay nor local effect, and the second one with $a = 0.6$, show that they have the same steady state. To compute them we took the same coefficients as in the same case for discrete delay, see (19).



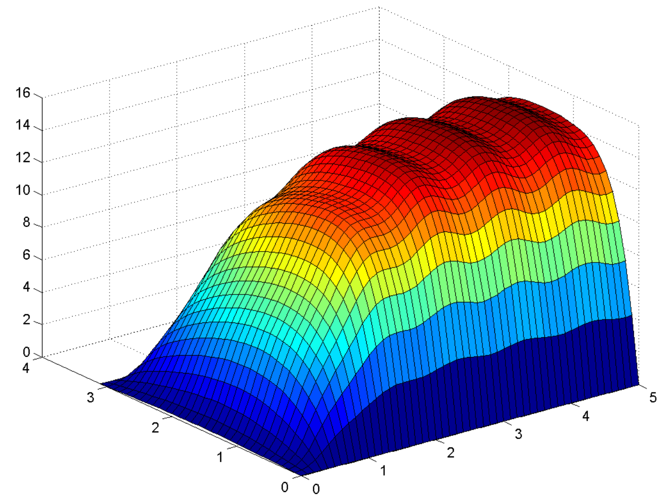
Parameter $a = 1.0$**, no delay.**



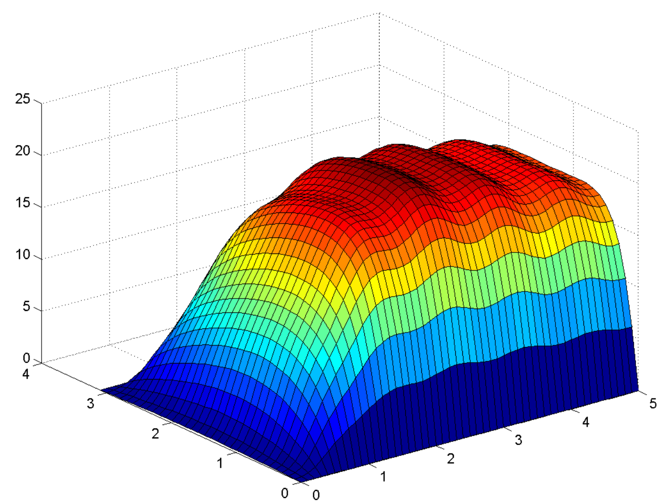
Parameter $a = 0.6$**, continuous delay.**

3.2.2 Periodic coefficients

For this case, the graphs seem to approximate to the same global attractor. They were computed with the same functions K, c, r, η as in the discrete delay case.



Parameter $a = 1.0$**, no delay.**



Parameter $a = 0.6$**, continuous delay.**

When the coefficients are almost periodic, the same behaviour was observed.

4 Conclusion.

As we mentioned before, [8], [9] and [10] studied the asymptotic behavior of equation (refequno) with and without delay. They proved for some particular cases for the coefficients, that as long as the weight of the instantaneous population is larger that that of the delayed population ($a > b$), the asymptotic behavior is the same, that is, they have the same steady state or the same global attractor. In [8] a discrete delay was considered without non-local effect and coefficients depending only on the space variable, in [9] they also work with discrete delay and no non-local effect but with coefficients depending on the x variable and periodic on t. Finally [10] considered continuous delay with non-local effect with constant coefficients. The examples we showed in 3.1.1 and 3.2 had not been considered yet and no analytical results have been obtained as the ones by Feng-Lu and Gourley-So. In all of these cases we notice that as long as the weight of the instantaneous populations is larger than the weight of the delay and non-local effect $(a > b)$, then both cases have the same asymptotic behavior. Based on the examples we have studied numerically, we expect that such result will hold in general for these three cases.

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