Equivalence of the periodic behavior in reversible one-dimensional cellular automata

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Abstract: - The periodic properties of reversible one-dimensional cellular automata are analyzed. The configuration set is taken as a topological space based on cylinder sets. Reversible one-dimensional cellular automata with neighborhood size 2 are used for representing the whole set of reversible automata. Using the characterization of reversible automata by block permutations, a matrix method is exposed for detecting the existence of fixed, periodic points; and non-wandering sets. Finally, we define a strong shift equivalence based on the periodic behavior.

Key-Words: - Cellular automata, block permutations, algorithms, dynamical systems

1 Introduction

One-dimensional cellular automata are discrete dynamical systems characterized by the simple interaction of their parts, but at the same time, these systems are able to produce complex global behaviors. This theory rises with the work developed by John von Neumann [10], he used these systems for constructing self-reproducing systems. Other relevant works are developed by John H. Conway [2] with the cellular automaton "Life" and by Stephen Wolfram [12] who analyzes the behavior of one-dimensional cellular automata.

The dynamical classification of cellular automata is up to now an open problem since there is not a general characterization of their behaviors. Nevertheless, in the case of reversible one-dimensional cellular automata such a characterization exists due to Jarkko Kari [4] who explains their behaviors by block permutations and shifts. Taking this characterization, this paper analyzes the periodic behavior of reversible one-dimensional cellular automata and establishes a strong shift equivalence.

The work has the following organization, section 2 gives the basic concepts about one-dimensional cellular automata, section 3 explains the behavior of the reversible one-dimensional cellular automata using block permutations and shifts. Section 4 provides the basic concepts for characterizing periodic systems, section 5 shows the periodical behavior of reversible automata using cylinder sets. With block permutations, the existence of fixed, periodic and non-wandering cylinder sets is proved. A connectivity relation is given for detecting some features of these behaviors and a classification of reversible automata is presented. Section 6 presents an example of these results and section 7 provides the conclusions of this work.

2 Basic concepts

A one-dimensional cellular automaton is formed by a set K of *states*, a sequence c of *cells*, where every cell takes one value of the set K. The cardinality of K is represented by k, and the sequence c will be called a *configuration* of the automaton. The set of all the possible configurations is the configuration set C. If configurations are infinite at both sides, every cell is indexed by an element of $\mathbb{Z}$, thus $c_{[i]}$ is the cell in position $i \in \mathbb{Z}$. For $n \in \mathbb{Z}^+$, notation K^n represents the set of sequences formed by states of K with length n, and notation K^* presents the set of all the finite sequences of states. For $i, j \in \mathbb{Z}$, $i < j$, $c_{[i,j]}$ is the sequence of states in c going from $c_{[i]}$ to $c_{[j]}$.

Given a cell in a configuration c, this one evolves into a new cell depending on its current state and the state of its r neighbors at each side. Thus r defines a *neighborhood radius*, and each cell and its neighbors form a *neighborhood* with $2r+1$ cells. Notation (k, r) represents a one-dimensional cellular automaton with k states and neighborhood radius r.

Each neighborhood form a new cell and this process is repeated over all the cells in a configuration. The mapping $\varphi : K^{2r+1} \to K$ from neighborhoods to states is the *evolution rule*. This rule forms a new configuration c', thus the evolution rule φ induces a global mapping Φ among the configurations of C.

A special kind of one-dimensional cellular automaton is the one which can return to previous stages of the system, in this case the evolution rule φ has an inverse rule φ^{-1} inducing a global mapping Φ^{-1} inverse to Φ. This kind of automata is *reversible*. Reversible cellular automata are relevant because they represent dynamical systems which conserve their initial information, for this reason they define a very interesting mathematical theory and have been used as models for data ciphering, information coding [5] and simulation of reversible physical phenomena [9, 12] among other applications.

2.1 Simulating a cellular automaton by another of neighborhood size 2

In a (k, r) cellular automaton, every neighborhood in K^{2r+1} forms a new cell applying the evolution rule. In other words, the ancestor sequence has $2r$ more cells than the successor one; if a sequence with $2r$ cells is taken, an ancestor of this one has $4r$ cells. Hence, for a sequence of $2r$ cells we can specify a partition of every ancestor in 2 sequences, each one with $2r$ cells (Figure 1).

So, a new set T is defined where every sequence of length $2r$ can be represented by a single state of the set T, and the cardinality of T is k^{2r}. Using T we have a new evolution rule τ simulating the original rule φ; the rule τ is a mapping from T^2 into T.

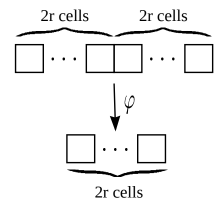

Fig. 1: Ancestor of a sequence of $2r$ cells, the ancestor is divided in 2 sequences, each of $2r$ cells.

By means of this process, a (k, r) cellular automaton may be simulated by another $(k^{2r}, 1/2)$ automaton. Of course, this procedure increases the number of states, nevertheless if a particular property in $(k, 1/2)$ automata is obtained, then this one is fulfilled for all kind of one-dimensional cellular automaton.

We are interested in the analysis of the periodic behavior produced by the global mappings of a $(k, 1/2)$ reversible automaton, for this reason we need a way to define distance and closeness among the configurations, therefore centered cylinder sets shall be used.

2.2 Centered cylinder sets

We must be able to handle finite sequences of states and indexing them in configurations. In order to yield subsets with similar configurations both for construction and for their periodic behavior, centered cylinder sets will be defined. Given a sequence $w \in K^*$ with odd length $| w |$, the following definition is presented:

Definition 1. *A centered cylinder set $\mathcal{C}_{[w]}$ is the set of configurations such that their central contiguous coordinates are equal to w, that is:*

$$\mathcal{C}_{[w]} = \left\{ c \mid c \in C, c_{[-|w|/2, |w|/2]} = w \right\} \tag{1}$$

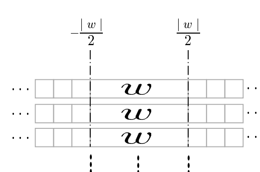

Fig. 2: Centered cylinder set specified by w.

The family $\mathfrak{C}$ is the family of all the centered cylinder sets in C. Using $\mathfrak{C}$, the configuration set C is compact, Hausdorff, and metric [3], [4] and C is a topological space represented by $(C, \mathfrak{C})$. An important feature of $\mathfrak{C}$ is that every centered cylinder set is disjoint to the others. For $w_1, w_2 \in K^*$, if w_1 contains w_2 as a central subsequence, then $\mathcal{C}_{[w_1]} \cap \mathcal{C}_{[w_2]} = \mathcal{C}_{[w_1]}$; but, if w_2 is not a centered subsequence of w_1, then $\mathcal{C}_{[w_1]} \cap \mathcal{C}_{[w_2]} = \varnothing$.

Taking advantage that $(C, \mathfrak{C})$ is compact, a finite covering can be always specified. In particular, for $n \in \mathbb{Z}^+$, n odd, take every sequence of the set K^n and form the family of centered cylinder sets specified by these sequences:

$$\mathfrak{C}_{K^n} = \left\{ \mathcal{C}_{[w]} \mid w \in K^n \right\} \tag{2}$$

This way, the family $\mathfrak{C}_{K^n}$ has k^n cylinder sets. Since every configuration belongs to a single centered cylinder set in $\mathfrak{C}_{K^n}$, then it covers the whole configuration set with a finite number of subsets. Every centered cylinder set can be considered as a set of nearby configurations, and different centered cylinder sets have distant configurations, so it is desirable to define a distance which gives a precise measure of the closeness among configurations. For configurations c and c', we establish the distance:

$$d(c, c') = \begin{cases} 0 & if \quad c = c' \\ \frac{1}{1+m} & if \quad c \neq c' \end{cases} \tag{3}$$

where $m \in \mathbb{N}$ and m is the minimum absolute value of the coordinate where $c_{[-m]} \neq c'_{[-m]}$ or $c_{[m]} \neq c'_{[m]}$. Thus every centered cylinder set $\mathcal{C}_{[w]}$ is a set of configurations where the maximum distance among its elements is $d = \frac{1}{1+m}$ where $m = \frac{|w|}{2} + 1$.

3 Characterization of reversible automata

The main result in the work of Kari [4] is that the action of every reversible one-dimensional cellular automaton can be represented by applying two block permutations and a shift. In order to explain the previous sentence, first the properties of reversible automata are explained.

3.1 Properties of reversible automata

Based on the work by Hedlund [3], the following properties are presented:

Remark 1. $(k, 1/2)$ *reversible automata have the following properties:*

1. *Every finite sequence of states have k ancestors.*
2. *For $n_0 \in \mathbb{Z}^+$ and $n \geq n_0$, the ancestors of every sequence in K^n have L left initial states, a single central part and R right final states, holding that $LR = k$.*

The first statement in Remark 1 is the uniform multiplicity of ancestors [6]; and values L and R in the second statement are Welch indices [3]. This way, a reversible automaton holds that every sequence has the same number of ancestors as the others, and the ancestors of each sequence have a common central part, leaving the differences into the ends.

3.2 Block permutations

For a reversible automaton with invertible rules φ and φ^{-1}, take the greatest value of the neighborhood size among both rules, and represent them with this size using redundant states if it is necessary. Since both rules have the same neighborhood size, we can simulate them by $(k, 1/2)$ reversible automata. Take this simulation, thus a neighborhood maps into a single state applying the inverse evolution rule φ^{-1}, but this neighborhood has k ancestors with the evolution rule φ, therefore a neighborhood has k ancestors which share a single common central state. This can be extended for larger sequences, for $n \geq 2$, a sequence of length n has k ancestors each one with $n+1$ cells, where the ancestors share a common sequence of length $n - 1$.

Take a sequence of 2 cells, this sequence has k ancestors of 3 cells, these ancestors have a common central cell. The same behavior exists for the inverse evolution rule φ^{-1} but in the inverse direction and with inverted Welch indices. So we can define two sets L_φ and R_φ, the elements of L_φ are sequences of 1 cell and one of its left ancestor cells (Figure 3). This is analogous for the elements of R_φ [4, 7].

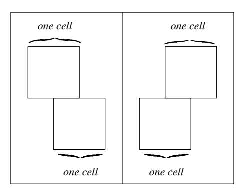

Fig. 3: Elements of L_φ and R_φ.

Thus L_φ has $\mid L_\varphi \mid = Lk$ elements and R_φ has $\mid R_\varphi \mid = Rk$ elements. Define two sets, X and Y, such that $\mid X \mid = \mid L_\varphi \mid$ and $\mid Y \mid = \mid R_\varphi \mid$, so we may specify a bijection both from L_φ into X and from R_φ into Y. The same process exists for the inverse rule φ^{-1}, there are bijections from $L_{\varphi^{-1}}$ into Y and from $R_{\varphi^{-1}}$ into X.

This way we have two block permutations p_1 and p_2, permutation p_1 goes from K^3 into all the possible sequences $x_i y_j$, where $x_i \in X$ and $y_j \in Y$ for $0 \leq i \leq Lk$ and $0 \leq j \leq Rk$. The second permutation p_2 is almost analogous, it goes from K^3 into the set of all the possible sequences $y_j x_i$. With these permutations, the evolution of a reversible automaton is represented by the composition $p_1 \circ p_2^{-1}$ of two block permutations and a shift 3/2 cells between both (Figure 4).

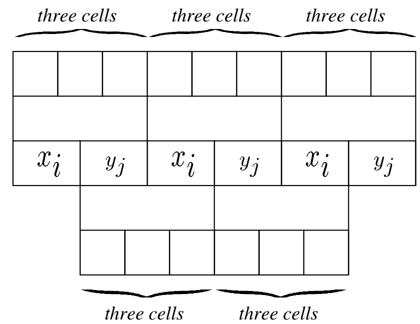

Fig. 4: Evolution of a reversible automaton represented by $p_1 \circ p_2^{-1}$.

4 Periodic behavior

The theory of dynamical systems consists of studying the long term behavior of a given system changing in time. The characterization of this behavior is obtained by knowing the conditions of the system such that it has a particular comportment. Using some topological concepts, a dynamical systems is defined in the following way:

Definition 2. *A dynamical system (X, Ψ) consists of a metric, compact space X and a continuous mapping $\Psi : X \to X$ which maps elements from X into itself.*

A consequence of Definition 2 is the orbit of a given point:

Definition 3. *In (X, Ψ), an orbit is the trajectory of a given point $x \in X$ yielded by the successive application of Ψ.*

The reason for using a compact space is that such spaces may be covered by a finite number of sets. With this, the orbit of x is described by the finite number of sets intersected by it. For the periodic behavior of a reversible automaton, we shall analyze the orbit of a periodic configuration c produced by Φ. Based on the works by J. de Vries [1] and Clark Robinson [8] some definitions of these behaviors are presented.

Definition 4. *A point $x \in (X, \Psi)$ is a periodic point with period n if $\Psi^n(x) = x$ and $\Psi^j(x) \neq x$ for $0 \leq j \leq n$*

If $x \in (X, \Psi)$ has period 1, then it is a *fixed* point. Another question is if there are orbits which come back not to the same point but to the same set; if this happens for every set in the covering of X, then we have the following definition:

Definition 5. *A dynamical system (X, Ψ) is nonwandering if for every set $\mathcal{O}$ in the covering of X there exists an integer $n > 0$ and a point $x \in \mathcal{O}$ such that $\Psi^n(x) \in \mathcal{O}$*

5 Analysis of the periodic behavior in reversible automata

In one-dimensional cellular automata, the global mapping Φ induced by φ generates the dynamical behavior in the configuration space $(C, \mathfrak{C})$. The orbit of any configuration is the progressive evolution produced by the iteration of the global mapping, and the sets visited by this orbit are the cylinder sets covering C.

We have not yet a complete characterization of one-dimensional cellular automata, and it is difficult to classify the evolution of a given automaton. However, for the reversible case this characterization exists and we will use it to define a matrix method for detecting distinct kinds of orbits. The $(k, 1/2)$ reversible automata shall be only studied; an orbit in cellular automata is defined as follows:

Definition 6. *For $i \in \mathbb{N}$ and configurations $c_i \in C$, an orbit $e = \{c_0, c_1, \ldots, c_i, \ldots\}$ is the sequence of configurations such that configuration c_{i+1} is the evolution of the configuration c_i*

Given an orbit e, its behavior is characterized by the intersected cylinder sets; in order of covering the configuration space $(C, \mathfrak{C})$, we use only the set K^3 because sequences of 3 cells are enough to form the block permutations. Block permutations define the transition from a cylinder set to another one, there is a shift among them of length 3/2 and the composition of these permutations is used for obtaining a shift of 3 cells. This process allows to work with centered cylinder sets.

In order to have a simpler notation, φ shall describe the composition and φ^{-1} will depict the composition in the inverse direction. For an orbit e, a mapping from a cylinder set $\mathcal{C}_{[c_{i[-1,1]}]}$ to $\mathcal{C}_{[c_{i+1[-1,1]}]}$ is defined in Figure 5.

5.1 Periodic behavior of $(k, 1/2)$ reversible automata

Suppose that a given configuration c is formed by the successive repetition of a finite sequence w of n cells,

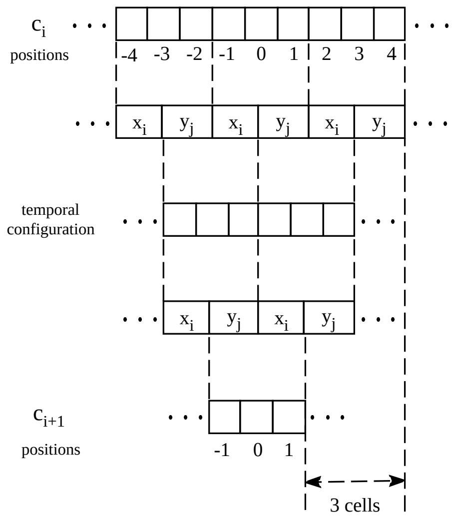

Fig. 5: Transition from $\mathcal{C}_{[c_{i[-1,1]}]}$ into $\mathcal{C}_{[c_{i+1[-1,1]}]}$.

the states forming this configuration have period n. Take an invertible evolution rule φ, since the action of this rule is a block permutation, the periodic global behavior is characterized by φ.

Theorem 1. *Given a $(k, 1/2)$ reversible automaton and a configuration c formed by the successive repetition of a finite sequence w of length n, the maximum period of the orbit produced by c is k^{3n}.*

Proof. Configuration c is formed by a periodic finite sequence w of n cells, take a sequence of w_1 of $3n$ cells in the configuration c. The sequence w_1 has also period $3n$ because is the repetition of a periodic sequence w of n cells. But w_1 also has n sequences of 3 cells each; applying the global mapping Φ induced by φ, every sequence of 3 cells maps into another single sequence of 3 cells by the block permutations. Thus the complete sequence w_1 of $3n$ cells maps into a single sequence w_2 of $3n$ cells as well.

Since all the sequences of $3n$ cells are in the finite set K^{3n}, and the cardinality of the set K^{3n} is $| K^{3n} | = k^{3n}$, then in some moment of the evolution the sequence w_1 is repeated. Thus the maximum period of the configuration c is k^{3n}. □

Periodic orbits e of period n induced by periodic configurations go from a centered cylinder set into the same one; then a consequence of Theorem 1 using Definition 5 is as follows:

Corollary 1. *For any sequence $w \in K^3$, the centered cylinder set $\mathcal{C}_{[w]}$ is a non-wandering set.*

Proof. Take any sequence w in K^3, and form a configuration c with the successive repetition of w. Then the configuration c belongs to the centered cylinder set $\mathcal{C}_{[w]}$ and is periodic. By Theorem 1, the orbit of c returns to $\mathcal{C}_{[w]}$ and therefore it is non-wandering □

5.2 Detecting some features of the periodic behavior

Transitions among block permutations can be used to find periodic orbits. Take the set K^3, with these sequences we can form k^3 configurations, each one obtained by the successive repetition of a single sequence in K^3.

Using block permutations, for $1 \leq i \leq k^3$, every configuration c_i has the form $\ldots w_i w_i w_i \ldots$ for $w_i \in K^3$ and every w_i maps into a single block $x_i y_i$ for $x_i \in X$ and $y_i \in Y$, and the whole configuration c_i maps into a sequence of blocks $\cdots x_i y_i x_i y_i x_i y_i \cdots$. Therefore there are only two kinds of blocks, $x_i y_i$ and $y_i x_i$; applying permutation p_2^{-1}, the block $y_i x_i$ maps into a single sequence $w_j \in K^3$. Making twice this process there is a mapping from $x_i y_i$ into another block $x_k y_k$ placed at the same coordinates (Figure 6).

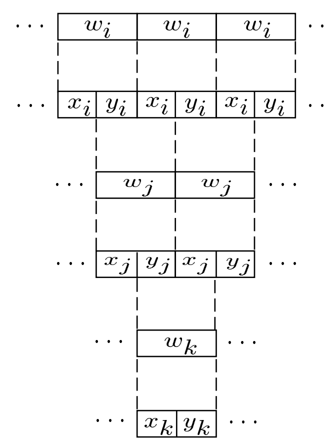

Fig. 6: Mapping from $x_i y_i$ into $x_k y_k$ placed at the same positions.

Applying the previous process to all the k^3 configurations, we have a bijective mapping among blocks xy because the reversibility of the automaton (every block has one ancestor and one successor). With this, we define a connectivity relation whose indices are blocks xy and the elements show the mapping from a given block into another (Table 1).

An equivalence relation may be defined in this connectivity relation; if $x_i y_i$ maps into $x_k y_k$ and it also maps into $x_m y_m$, then there is a mapping from $x_i y_i$ into $x_m y_m$. So, the transitive closure of the connectivity relation (calculated using Warshall's algorithm for instance) gets the equivalence relation. Due to the periodic behavior, every block xy returns to itself and the transitive closure is also reflexive. If there exists

xy				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
x_iy_i	$\vdots$	1	$\vdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
	$\cdots$	x_ky_k	$\cdots$	xy

Table 1: Connectivity relation defined by blocks xy.

a mapping from x_iy_i into x_my_m, then due also to the periodic behavior, there exists a mapping from x_my_m into x_iy_i and the relation is symmetric, therefore we have an equivalence relation. Every class in this relation represents a set of periodic configurations, the period of every class is its number of elements.

5.3 Classification based on the periodic behavior

The connectivity relation represents the mapping between centered cylinder sets defined by the orbit of periodic configurations. Since there is a finite number of cylinder sets, each one can reach a finite number of sets before returning to itself. In this sense the transitive closure of the connectivity relation is useful for comparing and classifying the periodic behavior of distinct reversible automata.

Definition 7. *Two reversible automata with evolution rules φ_1 and φ_2 belong to the same class if their connectivity relations have the same number of classes and there exists a bijection between classes with same periodicity. This equivalence will be denominated a periodic equivalence.*

5.4 Strong shift equivalence

Two shift systems represented by matrices A and B are strong shift equivalent if there exist matrices R and S such that:

$$A = RS \text{ and } B = SR. \quad (4)$$

If A and B are strong shift equivalent, the systems represented by them are topologically equivalent [11]. In this way, for two connectivity relations M_1 and M_2 if there exist two matrices R and S such that: $M_1 = RS$ and $M_2 = SR$ then the periodic behaviors of the respective reversible automata are topologically equivalent. One important problem is that there is not a general and easy process to find matrices R and S. However, this is not the case for connectivity relations M_1 and M_2. The following result is based on the work developed by Brian Marcus and Douglas Lind [5].

Theorem 2. *If two matrices M_1 and M_2 representing connectivity relations are periodically equivalent, then there are matrices R and S such that $M_1 = RS$ and $M_2 = SR$*

Proof. If M_1 and M_2 are periodically equivalent, each class in M_1 may be mapped to another class with the same form in M_2. For every class in M_1, map a centered cylinder set $\mathcal{C}_{[w_i]}$ into another centered cylinder set $\mathcal{C}_{[v_i]}$ of a similar class in M_2. Then, map $\mathcal{C}_{[v_i]}$ into $\mathcal{C}_{[w_j]}$ where $\mathcal{C}_{[w_i]}$ is communicated with $\mathcal{C}_{[w_j]}$ in M_1. Repeat the process, map $\mathcal{C}_{[w_j]}$ into $\mathcal{C}_{[v_j]}$ if $\mathcal{C}_{[v_i]}$ is connected with $\mathcal{C}_{[v_j]}$ in M_2. Applying iteratively this process for two similar classes in M_1 and M_2, every set in the class of M_1 maps into one and only one set the other class of M_2 and vice versa.

The mappings from M_1 into M_2 can be represented by a matrix R; and the mappings from M_2 into M_1 can be presented by another matrix S. Finally, the mapping defined between centered cylinder sets in M_1 can be also defined by the product RS, where R goes to the sets of M_2 and S returns to the sets of M_1. We have the same result for the mapping defined in M_2 by the product SR. Therefore $M_1 = RS$ and $M_2 = SR$.

□

A direct result of Theorem 2 is the following one:

Corollary 2. *If two reversible automata of neighborhood size 2 in both invertible rules are periodically equivalent, then their connectivity relations M_1 and M_2 are strong shift equivalent.*

6 Example

Two $(2, 1/2)$ reversible automata numbered by Wolfram's notation [12] are in Figure 7, the connectivity relations M_A and M_3 and their transitive closures are in Table 2.

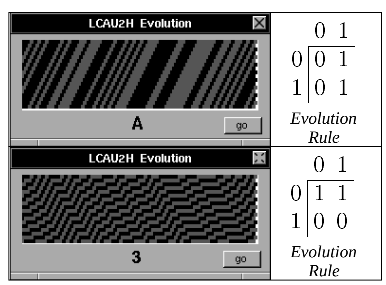

Fig. 7: $(2, 1/2)$ reversible one-dimensional cellular automata.

We have that both automata are periodically equivalent; take the class from M_A composed by

M_A

	000	001	010	011	100	101	110	111
000	1							
001			1					
010					1			
011							1	
100		1						
101				1				
110						1		
111								1

M_3

	000	001	010	011	100	101	110	111
000	1							
001					1			
010		1						
011						1		
100			1					
101							1	
110				1				
111								1

Closure of M_A

	000	001	010	100	011	101	110	111
000	1							
001		1	1	1				
010		1	1	1				
100		1	1	1				
011					1	1	1	
101					1	1	1	
110					1	1	1	
111								1

Closure of M_3

	000	001	010	100	011	101	110	111
000	1							
001		1	1	1				
010		1	1	1				
100		1	1	1				
011					1	1	1	
101					1	1	1	
110					1	1	1	
111								1

Table 2: Connectivity relations M_A and M_3 and their transitive closures.

$\{001, 010, 100\}$ and the class of M_3 composed by $\{011, 101, 110\}$. Map the sequence 001 into 011, and following the process of Theorem 2, the other mappings are presented in Figure 8.

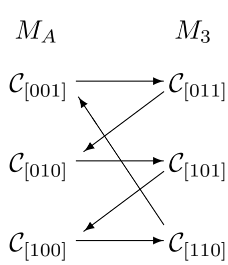

Fig. 8: Mappings defined by classes $\{001, 010, 100\} \subseteq M_A$ and $\{011, 101, 110\} \subseteq M_3$.

Applying the same process to the other classes, we yield the matrices R and S; R contains the mappings from M_A into M_3 and S describes the mappings from M_3 into M_A (Table 3). Thus $RS = M_A$ and $SR = M_3$ and both automata are strong shift equivalent with regard of the periodic behavior of their centered cylinder sets (Table 4).

R

	000	001	010	011	100	101	110	111
000								1
001				1				
010						1		
011			1					
100							1	
101					1			
110		1						
111	1							

S

	000	001	010	011	100	101	110	111
000								1
001						1		
010							1	
011			1					
100				1				
101					1			
110		1						
111	1							

Table 3: Matrices R and S.

$$\overset{R}{\begin{pmatrix} 00000001 \\ 00010000 \\ 00000100 \\ 00100000 \\ 00000010 \\ 00001000 \\ 01000000 \\ 10000000 \end{pmatrix}} \overset{S}{\begin{pmatrix} 00000001 \\ 00000100 \\ 00000010 \\ 00100000 \\ 00010000 \\ 00001000 \\ 01000000 \\ 10000000 \end{pmatrix}} = \overset{M_A}{\begin{pmatrix} 10000000 \\ 00100000 \\ 00001000 \\ 00000010 \\ 01000000 \\ 00010000 \\ 00000100 \\ 00000001 \end{pmatrix}}$$

$$\overset{S}{\begin{pmatrix} 00000001 \\ 00000100 \\ 00000010 \\ 00100000 \\ 00010000 \\ 00001000 \\ 01000000 \\ 10000000 \end{pmatrix}} \overset{R}{\begin{pmatrix} 00000001 \\ 00010000 \\ 00000100 \\ 00100000 \\ 00000010 \\ 00001000 \\ 01000000 \\ 10000000 \end{pmatrix}} = \overset{M_3}{\begin{pmatrix} 10000000 \\ 00001000 \\ 01000000 \\ 00000100 \\ 00100000 \\ 00000010 \\ 00010000 \\ 00000001 \end{pmatrix}}$$

Table 4: Products $RS = M_A$ and $SR = M_3$.

7 Conclusions

The topology of centered cylinder sets and block permutations provide a way for knowing the periodic behavior and classifying different reversible one-dimensional cellular automata. We have exposed a matrix method for studying the periodic behavior in these systems. The classification proposed in this paper is for automata whose invertible evolution rules have the same neighborhood size, and the representation of any automaton by another of neighborhood size 2 has been used for generalizing the process.

These methods are easily implemented in a computer when the number of states is small. Experimental observations show that Welch indices are not fundamental for establishing the class of a given reversible one-dimensional cellular automaton. For $(4, 1/2)$ reversible automata, a first examination provides a few periodic classes.

One extension of this work is analyzing transitive behaviors and establishing an equivalence between automata with distinct number of states.

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