Quantum Cohomology Ring for Hermitian Symmetric Spaces of type DIII

Yasunori Nishimori † Yoshihiro Ohnita††
†National Institute of Advanced Industrial Science and Technology, AIST
††Department of Mathematics, Tokyo Metropolitan University

Abstract: We determine the quantum cohomology ring for hermitian symmetric spaces of type DIII. Among the manifolds whose quantum cohomology ring has been rigorously computed, two hermitian symmetric spaces have been reported so far. The one we show in this paper, following a result of Sievert and Tian, is the third example. For this purpose encouting of the number of rational curves over the spaces satisfying a certain dimension condition is needed and it is accomplished by cell decomposition of hermitian symmetric spaces which are analogue of the Scubert cell for complex Grassmann manifolds.

Key-Words: quantum cohomology, Gromov-Witten invariant, intersection number, rational curve, cell decomposition, hermitian symmetric space

1 Introduction

The correlation functions of topological σ-model observed by Witten has a beautiful character that it satisfies a recursion relation for genus.

By Ruan [2] it is shown that this is interpretated as intersection number on moduli space of J-holomorphic curves and has been formulated in a mathematically rigorous way as invariants of semipositive symplectic manifolds. They are so called Gromov-Ruan-Witten invariants.

Moreover, Ruan-Tian [3] showed that the quantum cup product can be defined on cohomology $H^*(M, \mathbf{C})$ of a semi-positive symplectic manifold (M, ω) in terms of the invariants. The quantum cup product has such properties as non-graded, associative, anticommutative and so on, and it depends on the choice of Kähler class $[\omega]$ of M. Its homogeneous part, weak coupling limit $\lambda[\omega]$ for $\lambda \to \infty$, is equal to the ordinary cup product. We call quantum cohomology ring $H^*_{[\omega]}(M)$ a ring with a product structure on $H^*(M, \mathbf{C})$ defined by the quantum cup product.

However, as the delicate culculation to count the number of rational curves is involved, at present there are not so many examples of symplectic manifolds whose quantum cohomology ring are determined.

- complex Grassmann manifolds (Witten, Vafa, Siebert-Tian, Piunikhin etc.)
- complete flag manifolds associated to $U(n)$ (Gevental-Kim)
- partial flag manifold associated to $U(n)$ (Kim, Astashkevich-Sadov)
- toric manifolds (Batyrev)
- Calabi-Yau manifolds (Via Mirror symmetry by many physicists)
- complete intersection (Beauville)

- projective bundles on $\mathbf{C}P^n$ (Qin-Ruan)

From these results we see that the quantum cohomology ring is culculated only for two examples of Hermitian symmetric spaces of compact type at present : complex Grassmann manifolds $SU(m+n)/S(U(m)\times U(n))$, complex hyperquadrics $SO(n+2)/SO(n)\times SO(2)$. So in this paper, following a result of Sievert and Tian [4], we have determined the quantum cohomology ring of classical Hermitain symmetric spaces $SO(2n)/U(n)$ of type DIII as the third example. The result of Siebert and Tian says that if $H^*(M,\mathbf{C})$ is generated by $\alpha_1,\cdots,\alpha_s$ with relation $f^1,\cdots,f^t$, then $H^*_{[\omega]}(M,\mathbf{C})$ is a ring generated by $\alpha_1,\cdots,\alpha_s$ with new relation $f^1_\omega,\cdots,f^t_\omega$.

Theorem 1 The quantum cohomology ring of Hermitian symmetric space $SO(2n)/U(n)$ of type DIII is

$$H^*_{[\omega]}(SO(2n)/U(n)) \cong \mathbf{C}[e_2,e_4,\cdots,e_{2n-2}]$$
$$/(e_{4k}+\sum_{i=1}^{2k-1}(-1)^i e_{2i}e_{4k-2i}, e^2_{2n-2}-e^{-\lambda})$$

Here k runs from 1 to $n-2$.

Futhermore, if the corresponding σ-model has a description of Landau-Ginzburg type, such as in the case of complex Grassmann manifolds, then the Gromov-Witten invariant of higher genus can be expressed explicitly in terms of higer order residue integrals of potential functions, which is a formula of Vafa-Intriligator [4]. This can be considered as a kind of localization theorem, and we can expect to obtain an analogous result also in our case.

Proposition 1 For any $F\in\mathbf{C}[X_1,\cdots,X_k]$, the Gromov-Witten invariant of genus g for $SO(2n)/U(n)$ is given by

$$\langle F\rangle_g = c\sum_{dW^{[\omega]}=0}\det\left(\frac{\partial^2 W^{[\omega]}}{\partial X_i\partial X_j}\right)(\underline{x})F(\underline{x})$$

2 Gromov-Ruan-Witten invariants and the quantum cohomology ring

Let M be a $2n$-dimensional compact symplectic manifold with a symplectic form ω. (M,ω) is called positive (resp. semipositive) if, for any $R=f_*[S^2]\in H_2(M;\mathbf{Z})$ represented by $f:S^2\longrightarrow M$ with $[\omega](R)>0$, we have $c_1(M)(R)>0$ (resp. $c_1(M)(R)\geq 0$). We know that there exists an almost complex structure J on M tamed by ω, that is, $\omega(X,JX)>0$ for each nonzero $X\in TM$. Let Σ be a compact Riemann surface of genus g with the complex structure J. A smooth map $f:\Sigma\longrightarrow M$ is called a J-holomorphic curve if f satisfies the equations $J\circ df = df\circ j$, or equivalently $\bar\partial f=0$, where we define the Cauchy-Riemann operator $\bar\partial_J$ as $\bar\partial_J=\frac{1}{2}(df-J\circ df\circ j)$. For $\gamma\in C^\infty(T^*\Sigma\otimes f^{-1}TM)$, we consider its perturbed version $\bar\partial f=\gamma$ to define a perturbed or (J,γ)-holomorphic curve.

We recall the notion of the Gromov-Ruan-Witten invariant (GRW-invariant) ([3]). Let $R\in H_2(M;\mathbf{Z})$ and $[B_1],[B_2],\cdots,[B_s]\in H_*(M;\mathbf{Z})$. Here B_i $(i=1,2,\cdots,s)$ are pseudomanifolds.

The dimension condition is defined as

$$\sum_{i=1}(2n-\deg B_i)=2c_1(M)(R)+2n(1-g). \tag{dim}$$

If the dimension condition does not hold, then we define

$$\tilde\Phi_{(R,\omega)}([B_1],[B_2],\cdots,[B_s])=0.$$

If we assume that the dimension condition (dim) holds. For generic (J,γ), B_i $(i=1,\cdots,s)$, and $x_1,\cdots,x_s\in\Sigma$, the number of (J,γ)-holomorphic curves with $f(x_i)\in B_i$ $(i=1,\cdots,s)$ and $f_*[\Sigma]=R$ is finite. So we can define the number

$$\tilde{\Phi}_{(R,\omega)}([B_1],[B_2],\cdots,[B_s])$$

as the algebraic sum of such f with appropriate sign according to the orientation if the B_i $(i = 1,2,\cdots,s)$ are transversal to the Gromov boundary of the compactified moduli space of (J,γ)-holomorphic curves. It is known that this number $\tilde{\Phi}_{(R,\omega)}([B_1],[B_2],\cdots,[B_s])$ is independent of the choices of J,γ, points $x_1,\cdots,x_s \in \Sigma$, pseudo-manifolds $[B_1],[B_2],\cdots,[B_s]$, and the complex structure on Σ.

If J is an almost complex structure tamed by ω on M such that any J-holomorphic curve $f_*[\Sigma] = R$ is regular in the sense that the cokernel of the linearization operator of the Cauchy-Riemann operator $\bar{\partial}_J$ at f vanishes, then $(J,0)$ is generic. In the case where J is integrable and f is an immersion, the regularity at f is equivalent to the vanishing of $H^1(C;N_C)$, where $C = f(\Sigma)$ and N_C denotes the holomorphic normal bundle of C.

Let us consider the case where (M,ω) is a compact Kähler manifold and $R \in H_2(M;\mathsf{Z})$ such that any holomorphic curve C in M homologous to R is non-singular and has $H^1(C;N_C) = 0$. Suppose $B_1,\cdots,B_s$ be compact complex submanifolds in M transversal to the evaluation map and the Gromov boundary. Then

$$\tilde{\Phi}_{(R,\omega)}([B_1],\cdots,[B_s]) = \sum_C \sharp(B_1\cap C)\cdot\sharp(B_2\cap C)\cdot\cdots\cdot\sharp(B_s\cap C),$$

where the sum is taken over all holomorphic curves C homologous to R.

To define the quantum multiplication on $H^*(M,\mathsf{Z})$, we define

$$\tilde{\Phi}_{[\omega]}([B_1],\cdots,[B_s]) := \sum_{R\in H_2(M;\mathbb{Z})} \tilde{\Phi}_{(R,\omega)}([B_1],\cdots,[B_s])e^{-[\omega](R)}.$$

Suppose that (M,ω) is positive. We define the quantum cup product on $H^*(M;\mathsf{Z})$ by

$$(\alpha\wedge_Q\beta)[A] = \tilde{\Phi}_{[\omega]}(\alpha^V,\beta^V,A)$$

for each $A\in H_*(M;\mathsf{Z})$, where $\alpha,\beta\in H^*(M;\mathsf{Z})$ and α^V denotes the Poincaré dual of α. If we let $\{A_i\}$ a basis of the torsion free part of $H_*(M;\mathsf{Z})$ and $\{\alpha_i\}$ the Poincaré dual basis of $H^*(M;\mathsf{Z})$, then the quantum cup product can be expressed as

$$\alpha_i\wedge_Q\alpha_j = \sum_{k,l}\eta^{lk}\tilde{\Phi}_{[\omega]}(A_i,A_j,A_k)\alpha_l,$$

where (η^{lk}) is an inverse matrix to the intersection matrix $(\eta_{ij}) = (A_i\cdot A_j)$. Its homogeneous part reduces to the cup product

$$\alpha_i\wedge\alpha_j = \sum_{k,l}\eta^{lk}(A_i\cdot A_j\cdot A_k)\alpha_l.$$

We can describe the quantum cohomology ring by modifying generators and their relations of the classical cohomology ring.

Let $\mathsf{C}\langle X_1,\cdots,X_n\rangle$ be a graded anticommutative C-algebra defined by a relation $X_iX_j = (-1)^{d_id_j}X_jX_i$. Here X_i is an element of degree d_i. If m elements of $\{X_i\}$ are of odd degree, then $\mathsf{C}\langle X_1,\cdots,X_n\rangle$ is isomorphic to

$$(\Lambda^*\mathsf{C}^m)\otimes(\mathrm{Sym}^*\mathsf{C}^{n-m}).$$

An element of this C-algebra is called an ordered polynomial. Assume that (M,ω) is a compact symplectic manifold and its cohomology ring is expressed as

$$H^*(M,\mathsf{C}) = \mathsf{C}\langle X_1,\cdots,X_n\rangle/(f_1,\cdots,f_k),$$

where $f_i = \sum_{|J|=\deg(f_i)} a_{iJ}X^J$, $J = (j_1,\cdots,j_n)$, $X^J = X_1^{j_1}\wedge\cdots\wedge X_n^{j_n}$, $|J| = \sum_{i=1}^n j_id_i$. Here we assume that each $\deg(f_i)$ is even. We shall denote by $\hat{}$ an element of the quantum cohomology ring.

Lemma 1 [4] $\hat{X}_1, \cdots, \hat{X}_n$ generate the quantum cohomology ring $H^*_{[\omega]}(M; \mathsf{C})$.

Theorem 2 [4] The quantum cohomology ring for M is expressed as

$$H^*_{[\omega]}(M; \mathsf{C}) = \mathsf{C}\langle T_1, \cdots, T_n\rangle/(f_1^{[\omega]}, \cdots, f_k^{[\omega]}).$$

3 Quantum cohomology ring of $SO(2n)/U(n)$

It is known (cf.[6]) that the cohomology ring $H^*(SO(2n)/U(n); \mathsf{C})$ is described as follows :

$$H^*(SO(2n)/U(n), \mathsf{C}) \cong$$

$$\mathsf{C}[e_2, e_4, \cdots, e_{2n-2}]/(e_{4k} + \sum_{i=1}^{2k-1} (-1)^i e_{2i} e_{4k-2i}),$$

where $e_{2j} = 0$ for $j \geq n$.

The quantum cup product is defined by

$$\alpha_i \wedge_Q \alpha_j = \sum_{k,\ell} \eta^{\ell k} \tilde{\Phi}_{[\omega]}(A_i, A_j, A_k)\alpha_\ell,$$

where $\{A_i\}$ is a basis of the torsion free part for $H_*(M, \mathsf{Z})$.

We know that $H_2(SO(2n)/U(n); \mathsf{Z})$ is generated by a single class $[C]$ over Z, which is represented by a rational curve C of degree 1 as explained later. By the definition we have

$$\tilde{\Phi}_{d[C]}(\alpha^V, \beta^V, \gamma^V) = 0$$

unless

$$\deg \alpha^V + \deg \beta^V + \deg \gamma^V = 2c_1(M)(R) + 2\dim_{\mathbb{Z}} M.$$

All the cases where the dimension condition (dim) are satisfied are as follows :

- $d = 0$
- $d = 1, \quad k = n-1,$

namely

$$\deg \alpha^V + \deg \beta^V = 4(n-1), \deg \gamma^V = 2\dim_{\mathbb{C}} M.$$

Therefore we have only to determine the quantum product $e_{2n-2} \wedge_Q e_{2n-2}$. By the definition it becomes

$$e_{2n-2} \wedge_Q e_{2n-2} = \tilde{\Phi}_{[C]}(e^V_{2n-2}, e^V_{2n-2}, [*])e^{-[\omega](C)}.$$

We shall show that

$$\tilde{\Phi}_{[C]}(e^V_{2n-2}, e^V_{2n-2}, [*]) = 1.$$

Let $Gr_k(E)$ be the complex Grassmann manifold of all k-dimensional complex vector subspaces of a complex vector space E. We use the expression

$$\begin{aligned} SO(2n)/U(n) &= \{V \in Gr_k(\mathsf{C}^{2n}) \mid \mathsf{C}^{2n} = V \oplus \bar{V}\} \\ &= \{\text{orthogonal complex structures of } \mathsf{R}^{2n}\}. \end{aligned}$$

In the case where $n = 2$, $SO(4)/U(2) \cong \mathsf{C}P^1$. We assume that $n > 2$.

The rational curves C of degree 1 in $SO(2n)/U(n)$ are described as follows. Set

$$\begin{aligned} \mathcal{Z}_{n-2}(\mathsf{C}^{2n}) &= \{W \in Gr_{n-2}(\mathsf{C}^{2n}) \mid (W, W) = 0\}, \\ &= \frac{SO(2n)}{U(n-2) \times SO(4)}, \end{aligned}$$

where (,) denotes the standard symmetric complex bilinear form of C^{2n}. Hence the condition $(W, W) = 0$ means that W is perpendicular to $\bar{W}$ with respect to the standard Hermitian inner product of C^{2n}. Note that this space is a twistor space over the real Grassmann manifold $\tilde{Gr}_4(\mathsf{R}^{2n}) = \frac{SO(2n)}{SO(2n-4) \times SO(4)}$ of oriented 4-dimensional vector subspaces of R^{2n}.

The space $\mathcal{Z}_{n-2}(\mathsf{R}^{2n})$ parametrizes the set of all rational curves of degree 1 in $SO(2n)/U(n)$. We fix an arbitrary element $W \in \mathcal{Z}_{n-2}(\mathsf{C}^{2n})$. Using the W, we take an Hermitian orthogonal decomposition

$$\mathsf{C}^{2n} = W \oplus \bar{W} \oplus (W \oplus \bar{W})^\perp.$$

Set

$$\mathcal{Z}_2((W \oplus \bar{W})^{\perp}) = \{V_2 \in Gr_2((W \oplus \bar{W})^{\perp}) \mid (V_2, V_2) = 0\} \cong SO(4)/U(2).$$

Hence, for each $W \in \mathcal{Z}_{n-2}(\mathbf{C}^{2n})$, we get the canonical embedding

$$SO(4)/U(2) \longrightarrow \{W \oplus V \in SO(2n)/U(n) \mid V \in \mathcal{Z}_2((W \oplus \bar{W})^{\perp})\} \cong SO(4)/U(2),$$

which gives a rational curve $C = C_W$ of degree 1 in $SO(2n)/U(n)$. We can also express $C = C_W$ as

$$C_W = \{V \in SO(2n)/U(n) \mid W \subset V\}.$$

The complex hyperquadric is defined by

$$Q_{2n-2}(\mathbf{C}) = \{\ell \in \mathbf{C}P^{2n-1} \mid (\ell, \ell) = 0\}.$$

For each $\ell \in Q_{2n-2}(\mathbf{C})$, we set

$$B = B_\ell = \{W \in SO(2n)/U(n) \mid \ell \subset W\} \cong SO(2(n-1))/U(n-1).$$

Then we have (cf.[5])

$$[B_\ell] = e^V_{2n-2} \in H_{(n-1)(n-2)}(SO(2n)/U(n); \mathbf{C})$$

Now we fix $\ell_i \in Q_{2n-2}(\mathbf{Z})$ $(i = 1, 2)$, and set $B_1 = B_{\ell_1}$, $B_2 = B_{\ell_2}$, $[*] = V_3 \in SO(2n)/U(n)$. Then it follows $e^V_{2n-2} = [B_1] = [B_2]$.

Let Λ_i $(i = 1, 2, 3)$ be the intersection points of a rational curve $C = C_W$ with these three cycles $B_1, B_2, [*]$. We may assume that $\ell_1 \subset \Lambda_1 \supset U$ and $V_3 \not\supset \ell_1, \ell_2$. We have

$$\ell_2 \subset \Lambda_2 \supset W, \quad \ell_1, \ell_2 \not\supset W.$$

and

$$W_3 = \Lambda_3 \supset W, \quad \ell_i \cap W = \{0\}.$$

As $\Lambda_i \supset W$, we have $\Lambda_i \subset \bar{W}^{\perp}$. Thus we have $\Lambda_1 + \Lambda_2 + \Lambda_3 \subset \bar{W}^{\perp}$ and $\ell_1 \subset \Lambda_1$, $\ell_2 \subset \Lambda_2$, $V_3 = \Lambda_3$. Hence we see that if ℓ_1, ℓ_2, V_3 are linearly independent, then it follows $\Lambda_1 + \Lambda_2 + \Lambda_3 = \bar{W}^{\perp}$. In fact, we shall show that if ℓ_1, ℓ_2, W_3 are generic, then they become linearly independent. Since $\ell_2 + V_3$ is an $(n+1)$-dimensional complex subspace of $\mathbf{C}^{2n}$, $P(\ell_2 + V_3) \subset \mathbf{C}P^{2n-1}$ is defined as an n-dimensional complex projective subspace consisting of all 1-dimensional complex subspaces of $\ell_2 + V_3$. As $2n - 2 > n$, we have $P(\ell_2 + V_3) \cap Q_{2n-2}(\mathbf{C}) \subsetneq Q_{2n-2}(\mathbf{C})$. Thus we can choose an element $\ell_1 \in Q_{2n-2}(\mathbf{C}) \setminus \{P(\ell_2 + V_3) \cap Q_{2n-2}(\mathbf{C})\}$. Then ℓ_1 satisfies $\ell_1 \not\subset \ell_2 + V_3$ and hence ℓ_1, ℓ_2, V_3 are linearly independent. Since $W = \overline{(\ell_1 + \ell_2 + V_3)}^{\perp} \subset \bar{V}_3^{\perp} = V_3$, we have $W \perp \bar{W}$.

We choose a unique 1-dimensional vector subspace ℓ_0 of $\mathbf{C}^{2n}$ compatible with the standard orientation of $\mathbf{R}^{2n}$ such that

$$(\ell_1 \oplus W) \oplus \overline{(\ell_1 \oplus W)} \oplus (\ell_0 \oplus \bar{\ell}_0) = V_3 \oplus \bar{V}_3 = \mathbf{C}^{2n}.$$

Then we have

$$\begin{aligned} \Lambda_1 &= \ell_1 \oplus W \oplus \ell_0, \\ \Lambda_2 &= \ell_2 \oplus W \oplus \ell_0, \\ \Lambda_3 &= W_3. \end{aligned}$$

Therefore we obtain that the number of rational curves of degree 1 through B_1, B_2, $[*]$ is just one and the intersection number of the rational curve with each B_i $(i = 1, 2, 3)$ is one. It is easy to check that the transversality to the evaluation map and the Gromov boundary are satisfied in this case. We conclude that

$$\tilde{\Phi}_{[L]}(e^V_{2n-2}, e^V_{2n-2}, [*]) = 1.$$

From this we obtain the quantum cohomolgy ring for the space $SO(2n)/U(n)$:

$$H^*_{[\omega]}(SO(2n)/U(n)) \cong \mathbf{C}[e_2, e_4, \cdots, e_{2n-2}] \Big/ \Big(e_{4k} + \sum_{i=1}^{2k-1} (-1)^i e_{2i} e_{4k-2i}, e^2_{2n-2} - e^{-\lambda}\Big).$$

Here k runs from 1 to $n - 2$.

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