On the Problem of Distributing Objects in Cells with Different Capacities

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Abstract: - *A non-recursive and explicit expression for calculating the number of ways to distribute n indistinguishable objects into m distinguishable cells with different capacities has been proposed. Similar expression for the special case where cells have the same capacity was previously reported in the literature. Although the proposed expressions are interesting on their own in combinatorics subject, there are many applications for them, e.g. in studying network graphs with grid structures.*

Key-Words: - Combinatorics, Distributing objects in cells, Cells with different capacities.

1 Introduction

In this paper, we address a very famous problem in combinatorics: finding the number of ways to distribute n indistinguishable objects into m distinguishable cells where each cell has any arbitrary capacity. Despite its fame, however, there has not been, to our best knowledge, any study reporting an answer to this problem. The problem has widely been addressed for the special case where all cells have equal capacities.

In what follows, we first derive an explicit expression (Problem I) to calculate the number of ways to distribute n indistinguishable objects into m distinguishable cells where each cell may receive zero or any number of object up to its capacity. We then derive another expression (Problem II) generalising the first expression to calculate the number of ways to distribute n indistinguishable objects into m distinguishable cells where each cell receives at least a minimum predefined number of objects and up to as many objects as its capacity.

2 Problem I

In this section, we derive an expression to calculate the number of ways to distribute n indistinguishable objects into m distinguishable cells C_l, $1 \le l \le m,$ where each cell C_l has a capacity of $k_l - 1$ objects. Note that $n \le \sum_{l=l}^{m}(k_l - 1)$ since for $n > \sum_{l=l}^{m}(k_l - 1)$ the result is zero. One or more cells may have room for all n objects; we don't care which objects are in which cells, nor do we distinguish between positions in the cells; and cells need not be occupied.

The number of possible ways for such a distribution is given by the coefficient x^n [1-5] in polynomial $h(x)$

$$h(x) = \prod_{l=1}^{m}(1 + x + x^2 + ... + x^{k_l - 1}) = \prod_{l=1}^{m}(\frac{1 - x^{k_l}}{1 - x})$$

We can simply rewrite $h(x)$ as

$$h(x) = (1 + x + x^2 + ...)^m \prod_{l=1}^{m}(1 - x^{k_l}) = \left[\sum_{i=0}^{\infty}\binom{m+i-1}{m-1}x^i\right]\prod_{l=1}^{m}(1 - x^{k_l})$$

Let $g(x)$ be $\prod_{l=1}^{m}(1 - x^{k_l})$. In the expansion of $g(x)$, 2^m terms are produced that are correspondent to all 2^m combinations of *l*, $0 \le l \le m$, out of *m*. Each of these combinations can be constructed using a number J, $0 \le J \le 2^m - 1$. In binary representation of such an *m*-bit number, $J = j_m j_{m-1} ... j_1$, each bit position j_l, $1 \le l \le m$, is a placeholder for x^{k_l} in

$g(x)$, where a bit 1 in that position denotes x^{k_l} is present in the expanded term. As an example with m=5, the number J=19 with its corresponding bit string 10011, represents the term $x^{k_5}x^{k_2}x^{k_1} = x^{k_5+k_2+k_1} = x^{\sum_{l=1}^{m} j_l k_l}$. Moreover the number of 1's in the binary representation of each number J can be represented by $\sum_{l=1}^{m} j_l$. Using this representation we have

$$\prod_{l=1}^{m}(1-x^{k_l}) = \sum_{j=0}^{2^m-1}(-1)^{\sum_{l=1}^{m} j_l} x^{\sum_{l=1}^{m} j_l k_l}$$

For instance, the expansion of $g(x) = \prod_{l=1}^{3}(1-x^{k_l}) = (1-x^{k_3})(1-x^{k_2})(1-x^{k_1})$ is given by

$$g(x) = \sum_{j=0}^{7}(-1)^{\sum_{l=1}^{m} j_l} x^{\sum_{l=1}^{m} j_l k_l} = 1 - x^{k_1} - x^{k_2} + x^{k_1+k_2} - x^{k_3} + x^{k_1+k_3} + x^{k2+k_3} - x^{k_1+k_2+k_3}.$$

Expansion of $h(x)$ produces 2^m terms of the form

$$\left[\sum_{i=0}^{\infty}\binom{m+i-1}{m-1}x\right](-1)^{\sum_{l=1}^{m} j_l} x^{\sum_{l=1}^{m} j_l k_l} = (-1)^{\sum_{l=1}^{m} j_l}\sum_{i=0}^{\infty}\binom{m+i-1}{m-1}x^{i+\sum_{l=1}^{m} j_l k_l}$$

for J = 0, 1, …, 2^m-1. It follows that the coefficient of x^n in this term is equal to $\binom{m+n-1-\sum_{l=1}^{m} j_l k_l}{m-1}$. Summing up the coefficients of x^n in all the 2^m terms results in the desired expression as

$$\sum_{J=0}^{2^m-1}(-1)^{\sum_{l=1}^{m} j_l}\binom{m+n-1-\sum_{l=1}^{m} j_l k_l}{m-1}.$$

For the special case where all cells have the same capacity, the above equation can be simplified as follows. Advancing J between 0 and 2^m-1, make $\binom{m}{0}$ pattern with no bit "1", $\binom{m}{1}$ patterns with one bit "1", $\binom{m}{2}$ patterns with two bit "1", and so on. Thus, we can replace the sigma given above ($\sum_{J=0}^{2^m-1}(-1)^{\sum_{l=1}^{m} j_l}$) with the smaller sigma $\sum_{l=0}^{m}(-1)^l\binom{m}{l}$. Replacing $\sum_{l=1}^{m} j_l$ with l and $\sum_{l=1}^{m} j_l k_l$ with lk, we have a simpler equation as

$$\sum_{l=0}^{m}(-1)^l\binom{m}{l}\binom{n+m-lk-1}{m-1}$$

which was already reported in [1, 2, 4].

3 Problem II

Let us now derive an expression to calculate the number of ways to distribute n indistinguishable objects into m distinguishable cells C_l, $1 \le l \le m$, such that no cell C_l contains less than q_l and more than $q_l + k_l - 1$ objects.

Such an expression is given [1-5] by the coefficient of x^n in the polynomial $f(x)$

$$f(x) = \prod_{l=1}^{m}(x^{q_l} + x^{q_l+1} + ... + x^{q_l+k_l-1})$$

We can rewrite $f(x)$ as

$$f(x) = x^{\sum_{l=1}^{m} q_l}\prod_{l=1}^{m}(1+x+...+x^{k_l-1}) = x^{\sum_{l=1}^{m} q_l}h(x)$$

where $h(x)$ is exactly the polynomial given in problem I.

Recalling simple rules for generating functions [3-5], the coefficient of x^n in $f(x)$ can be easily obtained by coefficient $n - \sum_{l=1}^{m} q_l$ in $h(x)$. Thus we can derive the desired expression as

$$\sum_{j=0}^{2^m-1}(-1)^{\sum_{l=1}^{m} j_l}\binom{n+m-\sum_{l=1}^{m} q_l - 1 - \sum_{l=1}^{m} j_l k_l}{m-1}$$

For the special case where all the minimum object requirements for each cell is equal to q and the capacity of each cell is q+k-1, the above expression (using a similar simplification we used for Problem I) becomes

$$\sum_{l=0}^{m}(-1)^l\binom{m}{l}\binom{n+m-mq-lk-1}{m-1}$$

which was already reported in [1, 2].

4 Conclusions

We derived some non-recursive and explicit expression for calculating the number of ways to distribute n indistinguishable objects into m distinguishable cells each with a different minimum and maximum object limits. These expressions had already been derived for the special case where the minimum and maximum bounds for all cells are equal. Such expressions are very useful for studying topological properties of graphs with grid structures, e.g. calculating the number of nodes located at a given distance from a given node in the graph. These properties are very useful when, for example, studying the spanning tree of the grid topology, designing collective communication algorithms, and resource placement problem [6].

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