CONSTRUCTION OF α-VALUATIONS OF SPECIAL CLASSES OF 2-REGULAR GRAPHS

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This work is dedicated to the memory of Jaromir Abrham, a true gentleman, a scholar, and the inspiration for this work.

In this paper, we show that every 2-regular graph with three components of the form $2C_{4p} \cup C_{4m}$ has an α-labeling, except for the case p = m = 1. Furthermore, we present some general results for graphs composed of the disjoint union of cycles. The results considerably enlarge the class of 2-regular graphs known to have α-valuations.

Key-Words: Graph Labeling, α-valuation

1. BASIC DEFINITIONS

This paper is closely related to a companion paper by Eshghi, Carter & Abrham [3] where we show that all 2-regular graphs with three components of the form $C_{4a} \cup C_{4b} \cup C_{4c}$ have an α-labeling. Additional constructions are presented in Eshghi [4].

Let G = (V, E) be a graph with m = $|V|$ vertices and n = $|E|$ edges. By the term graph, we mean an undirected finite graph without loops or multiple edges.

A *graceful labeling* (or *b-valuation*) of a graph G = (V, E) is a one-to-one mapping Ψ of the vertex set V(G) into the set {0, 1, 2, … , n} with this property: If we define, for any edge e ={u, v}∈ E(G), the value $\Psi^\bullet(e) = |\Psi(u)-\Psi(v)|$ then $\Psi^\bullet$ is a one-to-one mapping of the set E(G) onto the set {1, 2, … , n}.

A graph is called graceful if it has a graceful labeling. An *a-labeling* (or *a-valuation*) of a graph G = (V, E) is a graceful labeling of G which satisfies the following additional condition: There exists a number γ ($0 \le \gamma \le |E(G)|$) such that, for any edge e

∈ E(G) with end vertices u, v ∈ V(G), $\min[\Psi(u), \Psi(v)] \le \gamma < \max[\Psi(u), \Psi(v)]$.

The concept of a graceful valuation and of an α-valuation were introduced by Rosa [8]. Rosa proved that, if G is graceful and if all vertices of G are of even degree, then $|E(G)| \equiv 0$ or 3 (mod 4). This implies that if G has an α-valuation and if all vertices of G are of even degree, then $|E(G)| \equiv 0$ (mod 4) (G is bipartite). In [8] it is also shown that these conditions are also sufficient if G is a cycle. The symbol C_m will denote a cycle on m vertices. Abrham and Kotzig [2] proved that Rosa's condition is also sufficient for 2-regular graphs with two components.

A snake is a tree with exactly two vertices of degree 1. In [8], it was proved that every snake has an α-valuation. A snake with n edges will be denoted by P_n.

A detailed history of the graph labeling problem and related results appears in Gallian [5, 6]. One of the

results of Abrham and Kotzig should be mentioned here: If G is a 2-regular graph on n vertices and n edges which has a graceful valuation Ψ then there exists exactly one number x ($0 < x < n$) such that $\Psi(v) \neq x$ for all $v \in V(G)$; this number x is referred to as the missing value of the graceful graph [2].
All parameters in this paper are positive integers. A sequence of numbers in parentheses or square brackets indicates the values of vertices of a graph or subgraph under consideration according to whether it is a snake or cycle respectively.

2. Transformations of Labeling of a Graph

The transformations presented below are used extensively in this paper.

Lemma 1: (Abrham & Kotzig [1]) Let r be a non-negative integer and let s be an odd integer, s = 2k+1 ∃2r+1. Then P_s has an α-valuation P with endpoints labelled w and z that satisfies the conditions z -w = k+1 and w = r. (w.l.o.g., we assume that w < z.)

Transformation 1: Given any 0 <= w <= k and k+1 # z # 2k+1, and z - w = k+1, we can always construct an α-valuation for a snake P_{2k+1} with edge labels 1 through 2k+1 and endpoints w and z, with (= k. i.e., the snake is bipartite. Since w = r, z = k+r+1.

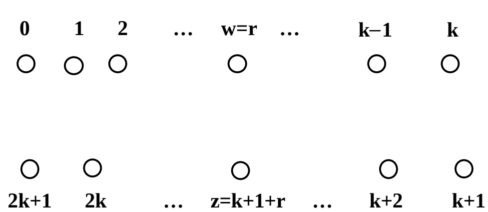

Figure 1: Arrangement of vertex labels of snake P_{2k+1} according to lemma 1

Suppose we now add n to the top half, and add m-(k+1) to the bottom half for any positive integers m and n where m > n+k:

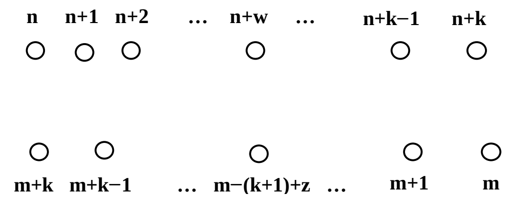

Figure 2: Arrangement of vertex labels in transformation 1

Then the edge labels will all increase by precisely m-(k+1)-n.The transformation produces the edge labels from [m-k–n] through [m+k–n].

Lemma 2: (Abrham & Kotzig [1]) Let r be a non-negative integer and let s be an even integer, s = 2k ∃4r+2. Then P_s has an α-valuation P with endpoints labelled w and z that satisfies the conditions z + w = k and w = r. (w.l.o.g., we assume that w < z.)

Transformation 2: Given any 0 # w < (k/2) < z # k and w + z = k we can always construct an α-valuation for a snake P_{2k} with endpoints w and z, with (= k. i.e., the snake is bipartite. Since w = r, z = k-r, and r < k/2.

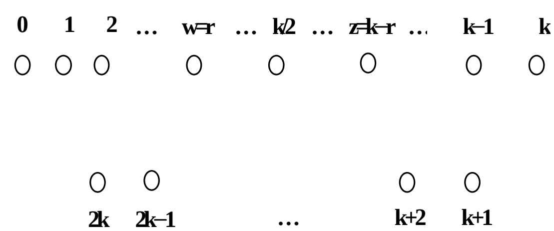

Figure 3: Arrangement of vertex labels of snake P_{2k} according to lemma 2

Suppose we now add n to the top half, and add m-(k+1) to the bottom half for any positive integers m and n where m > n+k:

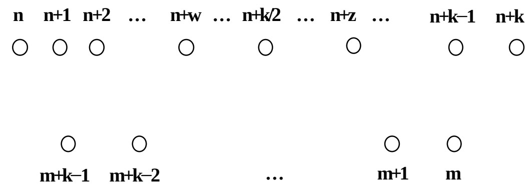

Figure 4: Arrangement of vertex labels in transformation 2

Then the edge labels will all increase by precisely m-(k+1)-n. The transformation produces the edge labels from [m-k–n] through [m+k-1– n].

Transformation 3: This transformation is derived similar to transformation 2. Given any α -valuation P of a graph on s edges, the *complementary valuation* is defined by substituting the labels of P(v) with s -P(v). The new valuation is again graceful. If we apply this transformation to the snake P_{2k} in transformation 2, we get:

Given any 2k ∃ w ∃ (3k/2) ∃z ∃ k and w + z = 3k, we can always construct an α-valuation for a snake P_{2k} with edge labels 1 through 2k and endpoints w and z, with (= k. i.e., the snake is bipartite. Since w = 2k-r, z = k+r, and 0 # r < k/2. (In the complement, w > z.)

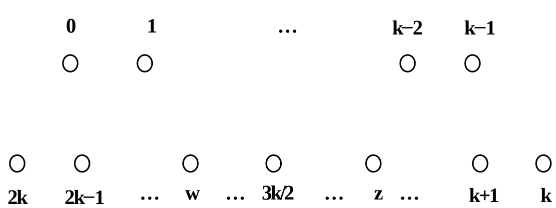

Figure 5: Arrangement of vertex labels of snake P_{2k} used in transformation 3

Suppose we add n to the top half, and add m-k to the bottom half for any positive integers m and n where m > n+k-1:

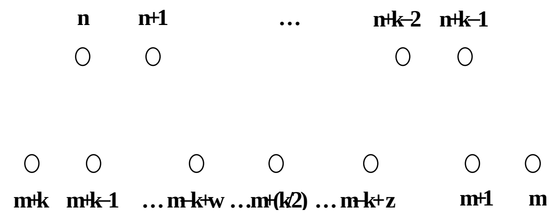

Figure 6: Arrangement of vertex labels in transformation 3

Then the edge labels will all increase by precisely m-k-n. The transformation produces the edge labels from [m-k–n-1] through [m+k– n].

3. The Construction of an α-valuation of the graph $2C_{4p} \cup C_{4m}$

Theorem 1: The graph $2C_{4p} \cup C_{4m}$ has an α-valuation for all m, p ≥ 1 with the exception of p = m = 1.

Proof: Since in [1] it was proved that if p, q ≥ 1 and p + q ≤ m then the graph $C_{4p} \cup C_{4q} \cup C_{4m}$ has an α-valuation, we only need to consider the case m < 2p in this theorem. We also know that $3C_4$ does not have an α-valuation [7]. Now we will organize the following cases, covering different special cases of this theorem, and prove each of them separately:

3.1 Case 1: p+2 < m < 2p

The vertices of the first C_{4p} will be successively labeled as follows: [0, 8p+4m, 1, 8p+4m-1, 2, 8p+4m-2, ... , p-1, 7p+4m+1, p+1, 7p+4m, ... , 2p-1, 6p+4m+2, 2p, 6p+4m+1]. The resulting edge values of the first C_{4p} are then 8p+4m, 8p+4m-1, 8p+4m-2, ... , 6p+4m+2, 6p+4m, ... , 4p+4m+2, 4p+4m+1 and 6p+4m+1.
The vertices of the second C_{4p} will be consecutively labeled by the numbers [2p+2m, 6p+2m, 2p+2m+1, 6p+2m-1, 2p+2m+2, 6p+2m-2, ... , 3p+2m-1, 5p+2m+1, 3p+2m+1, 5p+2m, ... , 4p+2m-1,

4p+2m+2, 4p+2m, 4p+2m+1]. The resulting edge values of the second C_{4p} are then: 1, 2, 3, ... , 2p-1, 2p, 2p+2, ... , 4p-1, 4p, 2p+1. The missing value of the first C_{4p} is equal to p and the missing value of the second C_{4p} is equal to 3p+2m. The missing value of the main graph is equal to 2p+m.

Now we must label the cycle C_{4m}. The cycle C_{4m} can be labeled based on the following stages:

i. Join the missing value of the first C_{4p}, i.e., p to the vertices labeled 5p+4m and 5p+4m-1. This generates the edges labeled 4p+4m and 4p+4m-1.

ii. Join the missing value of the second C_{4p}, i.e., 3p+2m to the vertices labeled 7p+2m+1 and 7p+2m+2. This generates the edges labeled 4p+1 and 4p+2.

iii. Construct the snake (6p+4m-1, 2p+1, 6p+4m-2, 2p+2, ... , 5p+4m+1, 3p-1, 5p+4m). Thus the edge labels 4p+4m-2, 4p+4m-3, 4p+4m-4, ... , 2p+4m+2, 2p+4m+1 will be generated by this snake.

iv. Form another snake in such a way that its vertices are labeled as follows: (5p+4m-1, 3p, 5p+4m-2, 3p+1, 5p+4m-3, ... , 2p+m-3, 6p+3m+1, 2p+m-2, 6p+3m).The value of the edges are then 4p+2m+2, 4p+2m+3, ... , 2p+4m-2, 2p+4m-1.

v. Join the vertex labeled 2p+2m-1 to the vertices labeled 6p+4m-1 and 6p+4m. The value of the edges will be 4p+2m and 4p+2m+1. Next join the two vertices 4p and 6p+4m and produce the edge labeled 2p+4m.

Now we have to distinguish ten special cases to cover the rest of the edge values of C_{4m} by considering this fact that the missing value of the main graph is equal to 2p+m. The details of construction of C_{4m} in each of these cases are given in Appendix 1.

3.2 Case 2: m = p + i   i = 2, 1, 0

The labeling of the vertices of the first and the second C_{4p} will be the same as case 1. Now we have to organize three special cases to label the edges of the cycle C_{4m}. The details of each case are given in Appendix 2.

3.3 Case 3: (1/2) p < m ≤ p-1

The vertices of the first C_{4p} will be successively labeled as follows: [0, 8p+4m, 1, 8p+4m-1, 2, 8p+4m-2, ... , p-1, 7p+4m+1, p+1, 7p+4m, ... , 2p-1, 6p+4m+2, 2p, 6p+4m+1]. The resulting edge values of the first C_{4p} are then 8p+4m, 8p+4m-1, 8p+4m-2, ... , 6p+4m+1, 6p+4m, ... , 4p+4m+2, 4p+4m+1. As we can see the first cycle C_{4p} is constructed exactly the same as the first cycle C_{4p} in the case 1.
The vertices of the C_{4m} will then be consecutively labeled by the numbers [4p, 4p+4m, 4p+1, 4p+4m-1, ... , 4p+m-1, 4p+3m+1, 4p+m+1, 4p+3m, ... , 4p+2m-1, 4p+2m+2, 4p+2m, 4p+2m+1]; this yields the edge values 4m, 4m-1, 4m-2, ... , 2m+2, 2m+1, 2m, 2m-1, ... , 3, 2, 1.

The second cycle C_{4p} can be labeled based on the following stages with the exception of case m = p-2:

i. The missing value of the first C_{4p}, p, is joined to the vertices 5p+4m and 5p+4m-1 to generate the edges labeled 4p+4m and 4p+4m-1.

ii. Join the missing value of C_{4m}, 4p+m, to the vertices labeled 4p+5m+2 and 4p+5m+1. This yields the edges labeled 4m and 4m+1.

iii. Form the following snake in such a way that its vertices are labeled by (6p+4m-1, 2p+1, 6p+4m-2, 2p+2, ... , 2p+m-2, 6p+3m+1, 2p+m-1, 6p+3m). The corresponding values of the edges are then 4p+4m-2, 4p+4m-3, 4p+4m-4, ... , 4p+2m+3, 4p+2m+2, 4p+2m+1.

iv. The edge labeled 4p+2m is obtained by connecting the vertex labeled 6p+4m to the vertex labeled 2p+2m.

v. The edges labeled 2p+4m+1 and 2p+4m are generated by joining the vertex labeled 4p-1 to the vertices labeled 6p+4m and 6p+4m-1 respectively.

vi. For (1/2)p < m ≤ p-3 form the snake (6p+3m, 2p+m+1, 6p+3m-1, 2p+m+2, … , 3p-2, 5p+4m+2, 3p-1, 5p+4m+1). The values of the edges of this snake are (4p+2m-1, 4p+2m-2, 4p+2m-3, … , 2p+4m+4, 2p+4m+3, 2p+4m+2).

Now we have to distinguish ten special cases to generate the remaining edge labels of C_{4p}. The details are given in Appendix 3.

3.4 Case 4: 1 < m ≤ (1/2) p

The vertices of the first C_{4p} will be successively labeled as follows: [0, 8p+4m, 1, 8p+4m-1, ...7p+4m+3, p-2, 7p+4m+2, p-1, 7p+4m+1, p, 7p+4m-1, ... , 2p-2, 6p+4m+1, 2p-1, 6p+4m]. The resulting edge values of the first C_{4p} are then 8p+4m, 8p+4m-1, 8p+4m-2, ... 4p+4m+3, 4p+4m+2, 4p+4m+1. The vertex labeled 7p+4m is the missing value of the first C_{4p}.

Suppose that m < (1/2)p. The vertices of C_{4m} will be consecutively labeled by the numbers [2p, 6p+4m-1, 2p+1, 6p+4m-2, …. 2p+m-2, 6p+3m+1, 2p+m-1, 6p+3m, 2p+m+1, 6p+3m-1, 2p+m+2, 6p+3m-2, … , 2p+2m-1, 6p+2m+1, 2p+2m, 6p+2m]; this yields the edge values 4p+4m-1, 4p+4m-2, 4p+4m-3, ... , 4p+2, 4p+1, 4p. The vertex labeled 2p+m is the missing value of $2C_{4p} \cup C_{4m}$.The construction of C_{4m} is shown in figure 11 as follows:

Now join the missing value of the first C_{4p} to the vertices labeled 3p and 3p+4m+1. This generates the edges labeled 4p+4m and 4p-1. Then we apply transformation type 2 to the vertex labels (2p+2m+1, 2p+2m+2, ... , 3p, ... , 3p+4m+1, … , 4p+2m-1, 4p+2m) and (4p+2m+1, 4p+2m+2, ... ,6p+2m-2, 6p+2m-1) by using the two vertices 3p and 3p+4m+1 as end vertices. Note that since 1 < m < (1/2) p we have 2p+2m+1 ≤ 3p < 3p+4m+1 ≤ 4p+2m. This transformation generates the edge labels 4p-2,4p-3, …., 3,2,1 and the construction of the second C_{4p} will be completed.

For m = (1/2)p the construction of C_{4m} and the second C_{4p} will be similar to the above case with a minor modification. The vertices of C_{4m} in this case will be labeled by the numbers [2p, 6p+4m-1, 2p+1, 6p+4m-2, …. , 2p+m-2, 6p+3m+1, 2p+m-1, 6p+3m, 2p+m+1, 6p+3m-1, 2p+m+2, 6p+3m-2, … , 2p+2m-1, 6p+2m+1, 2p+2m+1, 6p+2m]; this yields the edge values 4p+4m-1, 4p+4m-2, 4p+4m-3, ... , 4p+2, 4p, 4p-1. Now we connect the missing value of the first C_{4p}, i.e., 7p+4m to the vertices labeled 2p+2m and 3p+4m-1 (=5p-1) to generate the edges labeled 4p+4m and 4p+1. The edge labeled 4p-2 is obtained by joining the two vertices 2p+2m and 6p+2m-2. Next, we apply transformation type 1 to the vertex labels (2p+2m+2, 2p+2m+3, ... , 4p+2m-1, 4p+2m) and (4p+2m+1, 4p+2m+2, ... ,6p+2m-2, 6p+2m-1) by using the two vertices 4p+2m-1 (=5p-1) and 6p+2m-2 as end vertices. This transformation generates the edge labels 4p-3, 4p-4, …., 3,2,1 and the construction of the second C_{4p} will be completed.

3.5 Case 5: m = 1

As mentioned earlier, Abrham and Kotzig [1] showed that the case p=1 has no graceful valuation. The case p=2 was handled in case 4. Now suppose p > 3. The labeling of the vertices of the first C_{4p} will be successively as follows: [6p+5, 2p, 6p+6, 2p-1, ... , 7p+4, p+1, 7p+5, p-1, 7p+6, p-2, ... , 8p+3, 1, 8p+4, 0]. The edge labels of this cycle will be 4p+5, 4p+6, 4p+7, ... , 8p+2, 8p+3, 8p+4. The missing value of this cycle, p, will be used in the C_4.

The vertices of C_4 will be labeled as follows: [p, 5p+4, 3p+3, 5p+3]. The corresponding edge values of this cycle are then 4p+4, 2p+1, 2p, 4p+3. The missing value of the whole graph is 2p+1.

Now we will construct the second C_{4p}. First we generate a snake with vertices labeled (3p+2, 5p+5, 3p+1, 5p+6, ... , 2p+4, 6p+3, 2p+3, 6p+4, 2p+2). The resulting values of the edges are then 2p+3, 2p+4, 2p+5, ... , 4p-1, 4p, 4p+1, 4p+2. The edges labeled 2p+2 and 2p-1 are obtained by connecting the following pairs of vertices: 2p+2 and 4p+4; 3p+2 and 5p+1. In order to make the rest of the edge labels we will perform transformation type 3 to the vertex labels (5p+2, 5p+1, 5p, ... , 4p+5, 4p+4, 4p+3) and (4p+2, 4p+1, 4p, ... , 3p+5, 3p+4) where we select the two vertices 5p+1 and 4p+4 as end vertices. Therefore the edge labels 1, 2, 3, ... , 2p-3, 2p-2 will be obtained and the construction of the second C_{4p} will be completed.

For p = 3, we will have the graph $2C_{12} \cup C_4$ and an α-valuation of this graph could have the following vertex labels: [0, 23, 6, 24, 5, 25, 4, 26, 2, 27, 1, 28], [3, 19, 11, 18], [8, 22, 13, 15, 14, 17, 12, 16, 10, 20, 9, 21].

4. THE STANDARD VALUATIONS OF C_{4k}

Definition 1: The *standard α-valuations* of C_{4k} are given by any of the following sequence of values of the consecutive vertices of C_{4k}:

i) [4k, 0,4k-1, 1,4k-2, 2, … , k-2, 3k+1, k-1, 3k, k+1, 3k-1, k+2, 3k-2, … , 2k+2, 2k-1, 2k+1, 2k] with missing value x = k.

ii) [0,4k, 1,4k-1, 2, 4k-2,… , k-2, 3k+2, k-1, 3k-1, k+1, 3k, k+2, 3k-1, … , 2k-2, 2k+2, 2k, 2k+1] with missing value x = k.

iii) [4k, 0,4k-1, 1,4k-2, 2, … , k-2, 3k+1, k-1, 3k-1, k, 3k-2, … , 2k+1, 2k-2, 2k, 2k-1] with missing value x = 3k.

iv) [0,4k, 1,4k-1, 2, 4k-2,… , k-2, 3k+2, k-1, 3k+1, k, 3k-1, k+1, … , 2k-2, 2k+1, 2k-1, 2k] with missing value x = 3k.

In figure 7 one of the standard α-valuations of C_{12} has been shown:

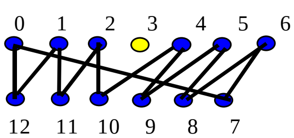

Figure 7: A standard α-valuation of C_{12}

A standard α-valuation of C_{4k} can be replaced by any other α-valuations of C_{4k}. For example, an α-valuation of C_{12} in figure 13 is replaced by an α-valuation of $2C_6$ in figure 8:

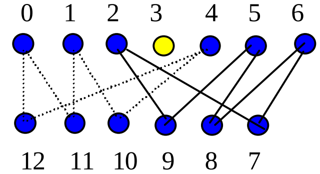

Figure 8: An α-valuation of $2C_6$

Definition 2: The graph C_{4k} has a *standard labeling* (or *standard valuation*) if the values of the vertices of C_{4k} can be generated from a standard α-labeling of C_{4k} differ by a constant factor.

For example, C_{12} in the α-labeling of $C_{12} \cup C_{20}$ shown in figure 9 has a standard labeling because it can be generated from a standard α-labeling of C_{12} that differs by a constant factor 10:

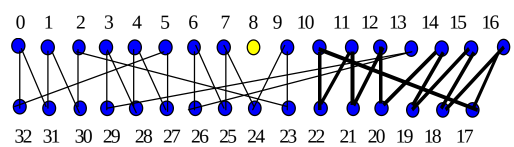

Figure 9: An α-valuation of $C_{12} \cup C_{20}$

If a graph has a standard labeling it can be replaced by any α-labeling of C_{4k} by considering the constant factor. For instance, the standard labeling of C_{12} in figure 10 can be replaced by an α-labeling of $2C_6$ to form an α-valuation of $2C_6 \cup C_{20}$ if we increase the values of the α-labeling $2C_6$ in figure 14 by constant factor i.e. 10:

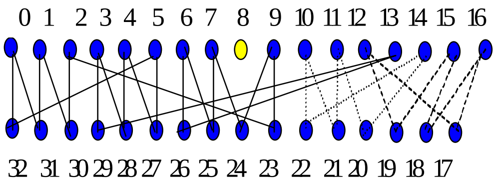

Figure 10: An α-valuation of C_{12}

5. EXISTENCE OF CONDITIONAL α-VALUATIONS OF GENERAL CLASSES OF 2-REGULAR GRAPHS

Now we present some general results for the graphs composed of the disjoint union of cycles. The results considerably enlarge the class of 2-regular graphs known to have α-valuations.

Theorem 2: The graph $\bigcup_{i=0}^{n} C_{4m_i}$ has an α-valuation if $\sum_{j=i+1}^{n} m_j \le m_i$ for i = 0, 1, 2, … , n-1.

Proof: According to the definition of standard labeling, we notice that in the construction of an α-valuation of the graph $C_{4p} \cup C_{4m}$; $p \le m$, which was constructed by Abrham and Kotzig [2], C_{4p} has a standard valuation. Suppose that $C_{4k} \cup C_{4m_0}$; $k \le m_0$, has an α-valuation. Now we replace C_{4k}, which has a standard labeling, by the graph $C_{4k_1} \cup C_{4m_1}$; $k_1 \le m_1$; $k = k_1 + m_1$. In this construction C_{4k_1} again has a standard valuation and we can replace it by $C_{4k_2} \cup C_{4m_2}$; $k_1 = k_2 + m_2$; $k_2 \le m_2$. If we repeat this kind of replacement in such a way that each time we replace a standard valuation of the graph C_{4k_i} by $C_{4k_{i+1}} \cup C_{4m_{i+1}}$; $k_i = m_{i+1} + k_{i+1}$; $k_{i+1} \le m_{i+1}$ for i = 2, … , n-2 and $k_{n-1} = m_n$; we will obtain an α-labeling of the graph $\bigcup_{i=0}^{n} C_{4m_i}$.

For example, the graph $C_{72} \cup C_{40} \cup C_{12} \cup C_8 \cup C_4$ has an α-valuation according to the theorem 2 since we have $m_0 = 18$, $m_1 = 10$, $m_2 = 3$, $m_3 = 2$ and $m_4 = 1$ and the condition of the theorem is satisfied.

Theorem 3: The graph $C_{4p} \cup C_{4r} \cup C_{4m}$ has an α-labeling where $C_{4m} = \bigcup_{i=1}^{n} C_{4m_i}$ and $p \ge r + \sum_{i=1}^{n} m_i$; $r = \sum_{i=2}^{n} m_i$; $\sum_{j=i+1}^{n} m_j \le m_i$ for i = 1, 2, … , n-1.

Proof: In the construction of an α-valuation of $C_{4k} \cup C_{4p}$; $k \le p$; we replace a standard labeling of C_{4k} by $C_{4m_1} \cup 2C_{4p_1}$; $k = m_1 + 2p_1$. We know that in an α-labeling of $C_{4m} \cup 2C_{4p}$ we are always able to construct at least one of C_{4p} by using a standard labeling; thus in an α-labeling of $C_{4m_1} \cup 2C_{4p_1}$ we can replace one of the C_{4p_1} by $C_{4p_2} \cup C_{4m_2}$; $p_1 = p_2 + m_2$; $p_2 \le m_2$. Now we use the same replacement procedure as theorem 2 in such a way that each time we replace a standard labeling of the graph C_{4p_i} by $C_{4p_{i+1}} \cup C_{4m_{i+1}}$; $p_i = m_{i+1} + p_{i+1}$; $p_{i+1} \le m_{i+1}$ for i = 2, … , n-2 and $p_{n-1} = m_n$.

For instance, the graph $C_{140} \cup C_{60} \cup C_{32} \cup C_{20} \cup C_8 \cup C_4$ has an α-valuation because the conditions of theorem 3 will be satisfied by assuming p = 35, r = 15, $m_1 = 8$, $m_2 = 5$, $m_3 = 2$ and $m_4 = 1$.

Theorem 4: The graph $C_{4s} \cup C_{4r} \cup C_{4m} \cup C_{4p}$ has an α-labeling where $C_{4m} = \bigcup_{i=1}^{n} C_{4m_i}$, $C_{4p} = \bigcup_{i=1}^{n} C_{4p_i}$ and $s \ge r + \sum_{i=1}^{n} (m_i + p_i)$; $r = p_n$ and $p_i = m_{i+1} + 2p_{i+1}$ for i = 1, 2, … , n-1.

Proof: In the construction of an α-valuation of $C_{4k} \cup C_{4p}$; $k \le p$; first we replace a standard labeling of C_{4k} by $C_{4m_1} \cup 2C_{4p_1}$; $k = m_1 + 2p_1$. Then we apply the replacement procedure by substituing a standard labeling of the graph C_{4p_i} by

$2C_{4p_{i+1}} \cup C_{4m_{i+1}}$; $p_i = m_{i+1} + 2p_{i+1}$ for i = 1,2, … , n-1.

For example, the graph $C_{200} \cup C_{16} \cup C_{80} \cup C_8 \cup C_{36} \cup C_{12} \cup C_{12} \cup C_4 \cup C_4$ has an α-valuation according to the theorem 4 if we assume $s = 50$, $r = 1$, $p_1 = 20$, $m_1 = 4$, $p_2 = 9$, $m_2 = 2$, $p_3 = m_3 = 3$ and $p_4 = m_4 = 1$.

Theorem 5: The graph $C_{4t} \cup C_{4s} \cup C_{4r} \cup C_{4m} \cup C_{4p}$ has an α-labeling where $C_{4m} = \bigcup_{i=1}^{n} C_{4m_i}$, $C_{4p} = \bigcup_{i=1}^{n} C_{4p_i}$ and $s = r + \sum_{i=1}^{n} (m_i + p_i)$; $r = p_n$ and $p_i = m_{i+1} + 2p_{i+1}$ for i = 1, 2, … , n-1.

Proof: First we consider an α-valuation of $C_{4t} \cup 2C_{4s}$. Then we replace a standard labeling of C_{4s} by $C_{4m_1} \cup 2C_{4p_1}$; $s = m_1 + 2p_1$. Next we apply the replacement procedure by substituting a standard labeling of the graph C_{4p_i} by $2C_{4p_{i+1}} \cup C_{4m_{i+1}}$; $p_i = m_{i+1} + 2p_{i+1}$ for i = 1,2, … , n-1.

Theorem 6: The graph $C_{4m} \cup C_{4s} \cup C_{4r}$ has an α-labeling where $C_{4m} = \bigcup_{i=0}^{n} C_{4m_i}$, $C_{4r} = \bigcup_{j=0}^{t} C_{4r_j}$, $s = \sum_{i=0}^{n} m_i$; $\sum_{j=i+1}^{n} m_j \le$ m_i for i = 0, 1, 2, … , n-1; $\sum_{l=j+1}^{t} r_l \le r_j$ for j = 0, 1, 2, … , t-1 ; $1 < \sum_{j=0}^{t} r_j < \sum_{i=0}^{n} m_i$ and $\sum_{j=0}^{t} r_j$ (1/2) $\sum_{i=0}^{n} m_i$.

Proof: We have seen that in construction of an α-valuation of $2C_{4p} \cup C_{4m}$ for $1< m < p$; m (1/2)p we are always able to construct at least one of C_{4p} and C_{4m} by using standard labelings; thus we use the same replacement procedure as theorem 2 for each of these graphs to obtain the an α-valuation of the graph $C_{4m} \cup C_{4s} \cup C_{4r}$.

Theorem 7: a) The graph $C_{4m} \cup 2C_{4s}$ has an α-labeling where $C_{4m} = \bigcup_{i=0}^{n} C_{4m_i}$; $s = \sum_{i=0}^{n} m_i$; $\sum_{j=i+1}^{n} m_j \le$ m_i for i = 0, 1, 2, … , n-1.

b) The graph $C_{4m} \cup C_{4s} \cup C_{4r}$ has an α-labeling where $C_{4m} = \bigcup_{i=0}^{n} C_{4m_i}$, $C_{4r} = \bigcup_{j=0}^{t} C_{4r_j}$, $s = \sum_{i=0}^{n} m_i$; $\sum_{j=i+1}^{n} m_j \le$ m_i for i = 0, 1, 2, … , n-1; $\sum_{l=j+1}^{t} r_l \le$ r_j for j = 0, 1, 2, … , t-1 and $\sum_{j=0}^{t} r_j = \sum_{i=0}^{n} m_i$.

Proof: In construction of C_{3a} , we have seen that two isomorphic components of C_{3a} have standard labelings. In part (a) of theorem 7 we use the same replacement procedure as theorem 2 for one of these components and in part (b) we use it for both of them. In fact in part b each standard labeling of C_{4a} decompose to the different components which are not necessarily isomorphic to each other.

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APPENDIX 1: Case 1: p+2 < m < 2p

Conditions: m = 2p-1, p ≥ 13

Step	The successive vertex labels
1	(4p-2,12p-5); (4p, 12p-8);(4p-2,10p-1)
2	(12p-3, 4p+1, 12p-4, 4p+2, 12p-5)
3	(11p-1, 5p-4, 11p-2, 5p-2, 11p-3, 5p-1, … , 10p+1, 6p-5, 10p, 6p-4, 10p-1)
4	(11p, 5p-5, 11p+1, 5p-3, 11p+4)
5	Apply the transformation type 3 to the vertex labels (12p-6, 12p-7, 12p-8, … , 11p+3, 11p+2) and (5p-6, 5p-7, 5p-8, … , 4p+4, 4p+3) by using the two vertices 12p-8 and 11p+4 as end points

Conditions: m = 2p-1, 4 ≤ p ≤ 12

p	m	The successive vertex labels of C_{4m}
12	23	[12, 152, 35, 153, 34, 154, 33, 155, 32, 156, 31, 157, 30, 158, 29, 159, 28, 160, 27, 161, 26, 162, 25, 163, 69, 164, 48, 136, 52, 137, 51, 138, 55, 134, 54, 135, 53, 131, 82, 132, 57, 133, 56, 130, 58, 129, 59, 128, 60, 127, 61, 126, 62, 125, 63, 124, 64, 123, 65, 122, 66, 121, 67, 120, 68, 119, 46, 139, 50, 140, 49, 141, 45, 142, 44, 143, 43, 144, 42, 145, 41, 146, 40, 147, 39, 148, 38, 149, 37, 150, 36, 151]
11	21	[11, 139, 32, 140, 31, 141, 30, 142, 29, 143, 28, 144, 27, 145, 26, 146, 25, 147, 24, 148, 23, 149, 63, 150, 44, 124, 48, 123, 49, 126, 47, 125, 52, 122, 50, 121, 75, 120, 51, 119, 53, 118, 54, 117, 55, 116, 56, 115, 57, 114, 58, 113, 59, 112, 60, 111, 61, 110, 62, 109, 42, 127, 46, 128, 45, 129, 41, 130, 40,131, 39, 132, 38, 133, 37, 134, 36, 135, 35, 136, 34, 137, 33, 138]
10	19	[10, 125, 30, 124, 31, 123, 32, 122, 33, 121, 34, 120, 35, 119, 36, 118, 37, 117, 41, 116, 42, 115, 38, 110, 68, 109, 46, 111, 45, 112, 44, 113, 43, 114, 52, 103, 53, 102, 54, 101, 55, 100, 56, 99, 47, 108, 48, 107, 49, 106, 50, 105, 51, 104, 40, 136, 57, 135, 21, 134, 22, 133, 23, 132, 24, 131, 25, 130, 26, 129, 27, 128, 28, 127, 29, 126]
9	17	[9, 112, 27, 111, 28, 110, 29, 109, 30, 108, 31, 107, 32, 106, 33, 105, 37, 104, 38, 103, 34, 98, 61, 99, 41, 100, 40, 101, 39, 102, 46, 94, 45, 95, 44, 96, 43, 97, 42, 89, 50, 90, 49, 91, 48, 92, 47, 93, 36, 122, 51, 121, 19, 120, 20, 119, 21,118, 22, 117, 23, 116, 24, 115, 25, 114, 26, 113]
8	15	[8, 99, 24, 98, 25, 97, 26, 96, 27, 95, 28, 94, 29, 93, 33, 92, 34, 91, 30, 86, 37, 87, 54, 88, 36, 89, 35, 90, 42, 81, 41, 82, 44, 79, 43, 80, 38, 85, 39, 84, 40, 83, 32, 108, 45, 107, 17, 106, 18, 105, 19, 104, 20, 103, 21, 102, 22, 101, 23, 100]
7	13	[7, 86, 21, 85, 22, 84, 23, 83, 24, 82, 25, 81, 29, 80, 30, 79, 26, 74, 33, 75, 32, 76, 47, 77, 31, 78, 38, 69, 37, 70, 36, 71, 35, 72, 34, 73, 28, 94, 39, 93, 15, 92, 16, 91, 17, 90, 18, 89, 19, 88, 20, 87]
6	11	[6, 73, 18, 72, 19, 71, 20, 70, 21, 69, 25, 68, 26, 67, 22, 62, 29, 61, 30, 60, 31, 59, 32, 66, 40, 65, 27, 64, 28, 63, 24, 80, 33, 79, 13, 78, 14, 77, 15, 76, 16, 75, 17, 74]
5	9	[5, 60, 15, 59, 16, 58, 17, 57, 21, 54, 33, 55, 18, 53, 22, 56, 24, 51, 23, 52, 26, 49, 25, 50, 20, 66, 27, 65, 11, 64, 12, 63, 13, 62, 14, 61]
4	7	[4, 47, 12, 46, 13, 45, 17, 44, 26, 43, 14, 40, 19, 39, 20, 42, 18, 41, 16, 52, 21, 51, 9, 50, 10, 49, 11, 48]

Conditions: (9/5) p+1 < m ≤ 2p-2 [Note: σ = 3m-5p-3, ω = 4p-2m+3, N = $\lfloor$σ/ ω$\rfloor$ -1, r = σ- (N+1) ω]

Step	The successive vertex labels
1	(6p+3m, 2p+m+1, 6p+3m-1, 2p+m+2, ... , 4p-1, 4p+4m+1, 4p+1, 4p+4m, ... , 6p-m, 2p+5m+1, 6p-m+1, 2p+5m) when m < 2p-2. For m = 2p-2 the vertex labels have the following order: (12p-6, 4p-1, 12p-7,

	4p+1, 12p-8, 4p+2, 12p-9, 4p+3, 12p-10)
2	(2p+m-1, 2p+5m); (2p+m-1,7m-2p-3);(11p+r+5, 5p-r);(4p,10p+4);(7p+2m+2+r, 9p-2m-r+4)
3	(7p+2m+r+2, p+2m-r-2, 7p+2m+r+1, p+2m-r-1, ... , 7p+2m+3, p+2m-3, 7p+2m+2) [Note: If r = 0, we need to exclude this snake from our consideration]
4	Apply the transformation type 3 on the vertex labels (6p-m+2,6p-m+3, … , p+2m-r-4, p+2m–r-3) and (7p+2m+r+3,7p+2m+r+4, …,2p+5m-2,2p+5m-1) by using the end vertices 11p+r+5 and 7m-2p-3.
5	(7p+2m+1, p+2m-2, 7p+2m, p+2m-1, ... , 3p+4m+r, 5p-r-1, 3p+4m+r-1, 5p-r)
6	Apply the transformation type 1 on the vertex labels (5p-r+1, 5p-r+2, ... , 9p-2m-r+4, ... , 2p+2m-3, 2p+2m-2) and (6p+2m+1, 6p+2m+2, ... , 10p+4, ... , 3p+4m+r-3, 3p+4m+r-2) by choosing the end vertices 9p-2m-r+4 and 10p+4.

Condition: m = (9/5)p+(3/5)	
Step	The successive vertex labels
1	(6p+3m, 2p+m+1, 6p+3m-1, 2p+m+2, ... , 4p-1, 4p+4m+1, 4p+1, 4p+4m, ... , 6p-m, 2p+5m+1, 6p-m+1, 2p+5m-1, ... , p+2m-3, 7p+2m+3, p+2m-2, 7p+2m+2)
2	(2p+m-1, 2p+5m); (2p+m-1, 7m-2p-2);(2p+5m, 5m-4p-3); (4p, 10p+2)
3	Apply transformation type 1 to the vertex labels (p+2m-1, p+2m, ... , 5m-4p-3, ... , 2p+2m-3, 2p+2m-2) and (6p+2m+1, 6p+2m+2, ... , 10p+2, ... , 7p+2m-1, 7p+2m) by using the two vertices 5m-4p-3 and 10p+2 as end vertices

Condition: m = (9/5)p+(4/5) [Note: The rest of the edge labels are generated by the same method as the above case]	
Step	The successive vertex labels
1	(6p+3m, 2p+m+1, 6p+3m-1, 2p+m+2, ... , 4p-1, 4p+4m+1, 4p+1, 4p+4m, ... , 6p-m, 2p+5m+1, 6p-m+1, 2p+5m-1, ... , p+2m-3, 7p+2m+3, p+2m-2, 7p+2m+1)
2	(2p+5m ,5m-4p-4);(2p+m-1, 7p+2m+2)
3	Apply the transformation type 1 to the vertex labels (p+2m-1**,** p+2m, …, 5m-4p-4, …, 2p+2m-3, 2p+2m-2) and (6p+2m+1, 6p+2m+2, … , 10p+2, … , 7p+2m-1, 7p+2m) by using the two vertices 5m-4p-4 and 10p+2 as end vertices.

Condition: m = (9/5)p+1 [Note: The rest of the edge labels are generated by the same method as the above case]	
Step	The successive vertex labels
1	(6p+3m, 2p+m+1, 6p+3m-1, 2p+m+2, … , 4p-1, 4p+4m+1, 4p+1, 4p+4m, … , 6p-m, 2p+5m+1, 6p-m+1, 2p+5m-1, … , p+2m-4, 7p+2m+4, p+2m-3, 7p+2m+2)
2	(2p+m-1 , 2p+5m);(2p+m-1, 7p+2m+3); (2p+5m,5m-4p-6);(7p+2m+3, p+2m-1); (4p, 10p}
3	(7p+2m+1, p+2m-2, 7p+2m, p+2m-1)
4	Apply the transformation type 1 to the vertices (p+2m, p+2m+1, ... , 5m-4p-6, ... , 2p+2m-3, 2p+2m-2) and (6p+2m+1, 6p+2m+2, ... , 10p, ... , 7p+2m-2, 7p+2m-1) by using the vertices 5m-4p-6 and 10p as end vertices.

Condition: (7/4) p+(1/2) < m ≤ (9/5) p+(2/5)	
Step	The successive vertex labels
1	(6p+3m, 2p+m+1, 6p+3m-1, 2p+m+2, ... , 4p-1, 4p+4m+1, 4p+1, 4p+4m, ... , 6p-m, 2p+5m+1, 6p-m+1, 2p+5m-1, ... , 7p+2m+3, p+2m-2, 7p+2m+2)
2	(2p+m-1,7m-2p-2); (2p+m-1, 2p+5m); (2p+5m,5m-4p-3); (4p, 10m-8p-4)
3	(7m-2p-2, 10p-3m+1, 7m-2p-1, 10p-3m, ... , 7p+2m, p+2m-1, 7p+2m+1)

4	Apply transformation type 1 to the vertex labels (10p-3m+2, 10p-3m+3, ... , 5m-4p-3, ... , 2m+2p-3, 2m+2p-2) and (6p+2m+1, 6p+2m+2, ... , 10m-8p-4, ... , 7m-2p-4, 7m-2p-3) and choose the two vertices 5m-4p-3 and 10m-8p-4 as end vertices

Conditions: (5/3) p +1 ≤ m ≤ (7/4) p+(1/2), m ≠ (22/13) p + (7/13)	
Step	The successive vertex labels
1	(6p+3m, 2p+m+1, 6p+3m-1, 2p+m+2, ... , 4p-1, 4p+4m+1, 4p+1, 4p+4m, ... , 6p-m, 2p+5m+1, 6p-m+1, 2p+5m-1, ... , 7p+2m+3, p+2m-2, 7p+2m+2)
2	(2p+m-1,2p+5m); (2p+5m, 5m-4p-3); (2p+m-1,7m-2p-2);(4p,20p-6m+5)
3	(5m-4p-3, 12p-m+3, 5m-4p-4, 12p-m+4, ... , p+2m, 7p+2m, p+2m-1, 7p+2m+1)
4	Apply transformation type 3 to the vertex labels (12p-m+2, 12p-m+1, ... , 7m-2p-2, ... , 20p-6m+5, ... , 6p+2m+2, 6p+2m+1) and (2p+2m-2, 2p+2m-3, ... , 5m-4p-1, 5m-4p-2) by using the two vertices 7m-2p-2 and 20p-6m+5 as end points
Conditions: (5/3) p +1 ≤ m ≤ (7/4) p+(1/2), m = (22/13) p + (7/13) [Note: The rest of the edge labels are generated by the same method as the above case]	
Step	The successive vertex labels
1	(2p+m-1,18p-5m+4); (4p , 6m-1)
2	Apply transformation type 3 to the vertex labels (12p-m+2, 12p-m+1, ... , 6m-1, ... , 18p-5m+4, ... , 6p+2m+2, 6p+2m+1) and (2p+2m-2, 2p+2m-3, ... , 5m-4p-1, 5m-4p-2) and select the two vertices 6m-1 and 18p-5m+4 as end vertices

Condition: m = (5/3)p+(1/3)	
Step	The successive vertex labels
1	(6p+3m, 2p+m+1, 6p+3m-1, 2p+m+2, ... , 4p-1, 4p+4m+1, 4p+1, 4p+4m, ... , 7p+2m+3, p+2m-1, 7p+2m+2)
2	(2p+m-1, 7p+2m+1); (2p+m-1,8p+m+1)
3	Apply transformation type 3 to the vertex labels (7p+2m, 7p+2m-1, ... , 10p+1, ... , 8p+m+1, ... , 6p+2m+2, 6p+2m+1) and (2p+2m-2, 2p+2m-3, ... , p+2m+1, p+2m) by using the two vertices 10p+1 and 8p+m+1 would be end vertices

Condition: m = (5/3)p+(2/3) [Note: The case where p = 5, m = 9 was discussed in m = 2p-1; so we can assume that the first values of p and m that satisfy the criteria are p = 8 and m = 14.]	
Step	The successive vertex labels
1	(6p+3m, 2p+m+1, 6p+3m-1, 2p+2m+2, … , 4p-1, 4p+4m+1, 4p+1, 4p+4m, … , p+2m-2, 7p+2m+3, p+2m-1, 7p+2m+1)
2	(2p+m-1, 7p+2m+2); (2p+m-1, 8p+m+2); (4p,10p+1)
3	Apply transformation type 3 to the vertex labels (7p+2m, 7p+2m-1, … , 10p+1, … , 8p+m+2, … , 6p+2m+1) and (2p+2m-2, 2p+2m-3, … , p+2m+1, p+2m) by choosing the vertices 10p+1 and 8p+m+2 as end vertices

Conditions: (3/2) p +1 < m ≤ (5/3) p, m ≠ (8/5)p + (2/5)	
Step	The successive vertex labels
1	(6p+3m, 2p+m+1, 6p+3m-1, 2p+m+2, 6p+3m-2, ... , 4p-1, 4p+4m+1, 4p+1, 4p+4m, 4p+2, ... , 7p+2m+4, p+2m-2, 7p+2m+3, p+2m-1, 7p+2m+2).
2	(2p+m-1, 2p+5m); (2p+m-1, 8p+m+1); (4p, 6m-1)
3	(7p+2m+1, p+2m, 7p+2m, p+2m+1, ... , 2p+5m+1, 6p-m, 2p+5m)

4	Apply transformation type 3 to the vertex labels (2p+5m-1, 2p+5m-2, ... , 6m-1, ... , 8p+m+1, ... , 6p+2m+2, 6p+2m+1) and (2p+2m-2, 2p+2m-3, ... , 6p-m+2, 6p-m+1) by selecting the vertices 6m-1 and 8p+m+1 as end vertices
Conditions: (3/2) p +1 < m ≤ (5/3) p, m =(8/5)p + (2/5) [Note: The first snake and the edge labeled 4m+1 will be obtained by the same procedure as case m ≠ (8/5)p + (2/5)]	
Step	The successive vertex labels
1	(4p, 10p+2); (2p+m-1, 6p+2m+1)
2	(7p+2m+1, p+2m, 7p+2m, p+2m-1, ... , 10p+4, 6p-m-1, 10p+3, 6p-m)
3	(6p-m, 10p+1, 6p-m+1, 10p, ... , 2p+2m-3, 6p+2m+2, 2p+2m-2, 6p+2m+1)

Conditions: m = (3/2) p + 1, p ≥ 8	
Step	The successive vertex labels
1	(6p+3m, 2p+m+1, 6p+3m-1, 2p+m+2, ... , 4p-1, 7p+2m+3, 4p+1, 7p+2m+2)
2	(2p+m-1, 2p+5m); (2p+m-1, 8p+m+1); (4p, 9p+5)
3	(10p+3, 4p+2, 10p+2, 4p+3, ... , (9/2)p-2, (19/2)p+6, (9/2)p-1, (19/2)p+5)
4	Apply transformation type 3 to the vertex labels ((19/2)p+4, (19/2)p+3, (19/2)p+2, ... , 9p+5, 9p+4, 9p+3) and (5p, 5p-1, ... , (9/2)p+1, (9/2)p) by using the two vertices (19/2)p+2 and 9p+5 as end vertices

Conditions: m = (3/2) p+1, p ≤ 6			
p	m	$2C_{4p} \cup C_{4m}$	An α-valuation of $2C_{4p} \cup C_{4m}$
4	7	$2C_{16} \cup C_{28}$	See case m = 2p-1, 4 ≤ p ≤ 12
6	10	$2C_{24} \cup C_{40}$	[0, 77, 12, 78, 11, 79, 10, 80, 9, 81, 8, 82, 7, 83, 5, 84, 4, 85, 3, 86, 2, 87, 1, 88], [32, 45, 44, 46, 43, 47, 42, 48, 41, 49, 40, 50, 39, 51, 37, 52, 36, 53, 35, 54, 34, 55, 33, 56], [6, 70, 17, 71, 16, 72, 15, 73, 14, 74, 13, 75, 31, 76, 24, 57, 30, 58, 29, 59, 28, 60, 21, 62, 26, 61, 27, 64, 38, 63, 25, 65, 23, 66, 20, 67, 19, 68, 18, 69]

Condition: m = (3/2) p + (1/2)	
Step	The successive vertex labels
1	((21/2)p+3/2, (7/2)p+3/2, (21/2) p+1/2, … , 4p-2, 10p+4, 4p-1, 10p+3)
2	((7/2)p-(1/2), (19/2)p+3/2); ((7/2)p-(1/2), (19/2)p+5/2); (4p, 9p+2)
3	(10p+2, 4p+1, 10p+1, 4p+2, … , (9/2)p-(3/2), (19/2)p+(7/2), (9/2)p-1/2, (19/2)p+5/2)
4	(10p+2, 6p+1, 10p+3)
5	((19/2)p+(3/2), (9/2)p+(1/2), (19/2)p+(1/2), … , 5p-2, 9p+3, 5p-1, 9p+2)

Conditions: (4/3) p+(2/3) < m ≤ (3/2) p, m ≠ (10/7) p + (4/7)	
Step	The successive vertex labels
1	(6p+3m, 2p+m+1, 6p+3m-1, 2p+m+2, ... , p+2m-3, 7p+2m+3, p+2m-2, 7p+2m+2)
2	(2p+m-1,8p+m+2); (2p+m-1,2p+5m-1); (4p,12p-2m+3)
3	(7p+2m+1, p+2m-1, 7p+2m, p+2m, ... , 4p-1, 4p+4m, 4p+1, 4p+4m-1, ... , 3m-3, 8p+m+3, 3m-2, 8p+m+2)
4	Apply transformation type 3 to the vertex labels (8p+m+1, 8p+m, ... , 2p+5m-1, ... , 12p-2m+3, ... , 6p+2m+2, 6p+2m+1) and (2p+2m-2, 2p+2m-3, ... , 3m, 3m-1) by selecting the two vertices 2p+5m-1 and 12p-2m+3 as end vertices
Conditions: (4/3) p+(2/3) < m ≤ (3/2) p, m = (10/7) p + (4/7)	
Step	The successive vertex labels
1	(6p+3m, 2p+m+1, 6p+3m-1, 2p+m+2, ... , p+2m-3, 7p+2m+3, p+2m-2, 7p+2m+2)
2	(2p+m-1,8p+m+2); (2p+m-1, 8p+m+1); (7p-2m+2, 7p+2m+1); (13p-6m+4,7p+2m); (7p-2m+2, 15p-4m+5); (4p , 26p-12m+10)
3	(7p+2m, p+2m-1, 7p+2m-1, p+2m, ... , 4p+4m, 4p-1, 4p+4m-1, 4p+1, ... , p+6m-1, 7p-2m+1, p+6m-2, 7p-2m+3, ... , 3m-3, 8p+m+3, 3m-2, 8p+m+2)
4	(8p+m+1, 3m-1, 8p+m, 3m, ... , 8m-7p-7, 15p-4m+6, 8m-7p-6, 15p-4m+5)

5	Apply transformation type 1 to the vertex labels (8m -7p-5, 8m-7p-4, ... , 2p+2m-3, 2p+2m-2) and (6p+2m+1, 6p+2m+2, ... , 15p-4m+3, 15p-4m+4) by choosing the vertices 8m-7p-4 and 6p+2m+2 as end points

Condition : p+3 ≤ m ≤ (4/3)p+(2/3)	
Step	The successive vertex labels
1	(6p+3m, 2p+m+1, 6p+3m-1, 2p+m+2, ... , p+2m-3, 7p+2m+3, p+2m-2, 7p+2m+2)
2	(7p+2m+1, p+2m-1, 7p+2m, p+2m, ... , 3m-4, 8p+m+3, 3m-3, 8p+m+2); (2p+m-1,10p-m+3)
3	Apply transformation type 1 to the vertex labels (3m-2, 3m-1, ... , 4p, ... , 2p+2m-3, 2p+2m-2) and (6p+2m+1, 6p+2m+2, ... , 10p-m+3, ... , 8p+m, 8p+m+1) by selecting the two end vertices 4p and 10p-m+3

Appendix 2: Case 2: m = p + i i = 2, 1, 0

Condition : m = p + 2 , p ≥ 4	
Step	The successive vertex labels
1	(p, 9p+7) ; (p, 9p+8); (3p+2m, 9p+5); (3p+2m, 9p+6); (4p,10p+8); (4p+3, 10p+8); (4p+3,10p+7); (3p+1,9p+4); (3p+3,9p+5); (3p+3,9p+4);(3p+1,9p+1)
2	(10p+7, 2p+1, 10p+6, 2p+2, ... , 3p-2, 9p+9, 3p-1, 9p+8)
3	(9p+7, 3p, 9p+6)
4	Apply transformation type 1 to the vertex labels (3p+4, 3p+5, ... , 4p, 4p+1, 4p+2) and (8p+5, 8p+6, ... , 9p+1, 9p+2, 9p+3) by choosing the two vertices 4p and 9p+1 as end points

Condition : m = p + 2 , p = 1,2,3

p	m	$2C_{4p} \cup C_{4m}$	An α-valuation of the graph $2C_{4p} \cup C_{4m}$
3	5	$2C_{12} \cup C_{20}$	[0, 44, 1, 43, 2, 42, 4, 41, 5, 40, 6, 39], [16, 28, 17, 27, 18, 26, 20, 25, 21, 24, 22, 23], [3, 34, 9, 33, 19, 32, 14, 29, 13, 30, 10, 31, 12, 38, 15, 37, 7, 36, 8, 35]
2	4	$2C_8 \cup C_{16}$	[0, 29, 4, 30, 3, 31, 1, 32], [12, 19, 13, 18, 15, 17, 16, 20], [2, 26, 6, 21, 10, 22, 9, 23, 14, 24, 5, 27, 11, 28, 7, 25]
1	3	$2C_4 \cup C_{12}$	[0, 19, 2, 20], [8, 11, 10, 12], [1, 16, 3, 15, 9, 14, 6, 13, 4, 18, 7, 17]

Conditions : m = p + 1 , p ≥ 10	
Step	The successive vertex labels
1	(p,9p+4); (p, 9p+3); (5p+2 , 9p+3); (4p+1,8p+3); (5p+2,9p+5); (2p+1, 8p+6); (4p,10p+4);(4p+1,10p+4); (4p-6, 8p+3);
2	(2p+1, 10p+3, 2p+2, 10p+2, ... , 3p-2, 9p+6, 3p-1, 9p+5)
3	(4p, 8p+4, 4p-1, 8p+5, 4p-2, 8p+6)
4	(9p-3, 3p, 9p+1, 3p+3, 9p+2, 3p+2, 9p+4)
5	Apply transformation type 1 to the vertex labels (3p+4, 3p+5, ... , 4p-6, 4p-5, 4p-4, 4p-3) and (8p+7, 8p+8, ... , 9p-3, 9p-2, 9p-1, 9p) by selecting the two vertices 4p-6 and 9p-3 as end points

Conditions : m = p + 1 , 1 ≤ p ≤ 9

p	m	C_{4m}	The labeling of the cycle C_{4m}
9	10	C_{40}	[9, 84, 47, 86, 26, 87, 25, 88, 24, 89, 23, 90, 22, 91, 21, 92, 20, 93, 19, 78, 33, 80, 32, 81, 31, 82, 29, 83, 27, 79, 35, 77, 34, 75, 37, 94, 36, 76, 30, 85]
8	9	C_{36}	[8, 75, 42, 77, 23, 78, 22, 79, 21, 80, 20, 81, 19, 82, 18, 83, 17, 70, 30, 71, 29, 72, 28, 67, 33, 84, 32, 68, 31, 69, 24, 73, 27, 74, 26, 76]
7	8	C_{32}	[7, 66, 37, 68, 20, 69, 19, 70, 18, 71, 17, 72, 16, 73, 15, 62, 27, 61, 25, 63, 26, 59, 29, 74, 28, 60, 21, 64, 24, 65,

			23, 67]
6	7	C_{28}	[6, 57, 32, 59, 17, 60, 16, 61, 15, 62, 14, 63, 13, 54, 22, 53, 23, 52, 24, 64, 25, 51, 18, 55, 21, 56, 20, 58]
5	6	C_{24}	[5, 48, 27, 50, 14, 51, 13, 52, 12, 53, 11, 46, 15, 43, 21, 54, 20, 44, 19, 45, 18, 47, 17, 49]
4	5	C_{20}	[4, 39, 22, 41, 11, 42, 10, 43, 9, 38, 15, 37, 12, 36, 16, 44, 17, 35, 14, 40]
3	4	C_{16}	[3, 30, 17, 32, 7, 33, 9, 29, 11, 28, 12, 34, 13, 27, 8, 31]
2	3	C_{12}	[2, 21, 12, 23, 6, 24, 8, 20, 5, 19, 9, 22]
1	2	C_8	[1, 12, 7, 14, 5, 11, 3, 13]

Condition : m = p [An α-valuation of 3 C_{4p} was constructed by Abrham and Kotzig in [1].

Appendix 3 : Case 3: (1/2) p < m ≤ p-1

Conditions : m = p-1

Step	The successive vertex labels
1	(9p-5, 3p, 9p-6, 3p+1, ... , 8p-2, 4p-3, 8p-3, 4p-2) [Note: For p = 2 and m = 1 an example of an α-valuation of $2C_8 \cup C_4$ is as follows: [0, 17, 4, 18, 3, 19, 1, 20], [2, 13, 6, 16, 7, 15, 9, 14], [8, 11, 10, 12]. The missing value is 5]

Conditions : m = p-2, p ≥ 13

Step	The successive vertex labels
1	(p, 9p-8); (2p+1, 10p-8); (p, 9p-10); (5p-2, 9p-9); (5p-2, 9p-8); (4p-2,10p-12); (4p-1,10p-9); (4p-1,10p-8); (2p+1,10p-16);
2	(4p-2, 8p-7, 4p-3, 8p-6, ... , 3p+2, 9p-11, 3p+1, 9p-10)
3	(9p-9, 3p, 9p-4, 3p-4, 9p-5, 3p-3, 9p-6, 3p-1, 9p-7, 3p-8)
4	(10p-12, 2p+4, 10p-11, 2p+3, 10p-10, 2p+2, 10p-9)
5	Apply transformation type 1 to the vertex labels: (2p+5, 2p+6, ... , 3p-8, 3p-7, 3p-6, 3p-5) and (9p-3, 9p-2, ... , 10p-16, 10p-15, 10p-14, 10p-13) by using the two vertices 3p-8 and 10p-16 as end points.

Conditions : m = p-2, 3 ≤ p ≤ 12

p	C_{4p}	The labeling of C_{4p}
12	C_{48}	[12, 100, 58, 99, 36, 105, 31, 106, 30, 107, 29, 102, 35, 101, 33, 103, 32, 104, 25, 112, 47, 111, 26, 110, 27, 109, 28, 108, 46, 89, 45, 90, 44, 91, 43, 92, 42, 93, 41, 94, 40, 95, 39, 96, 38, 97, 37, 98]
11	C_{44}	[11, 91, 53, 90, 33, 96, 29, 95, 27, 97, 28, 93, 32, 92, 30, 94, 23, 102, 43, 101, 24, 100, 25, 99, 26, 98, 42, 81, 41, 82, 40, 83, 39, 84, 38, 85, 37, 86, 36, 87, 35, 88, 34, 89]
10	C_{40}	[10, 82, 48, 81, 30, 87, 25, 86, 26, 85, 27, 83, 29, 84, 21, 92, 39, 91, 22, 90, 23, 89, 24, 88, 38, 73, 37, 74, 36, 75, 35, 76, 34, 77, 33, 78, 32, 79, 31, 80]
9	C_{36}	[9, 73, 43, 72, 27, 77, 23, 76, 24, 75, 26, 74, 19, 82, 35, 81, 20, 80, 21, 79, 22, 78, 34, 65, 33, 66, 32, 67, 31, 68, 30, 69, 29, 70, 28, 71]
8	C_{32}	[8, 64, 38, 63, 24, 69, 19, 67, 20, 66, 23, 65, 21, 70, 18, 71, 31, 72, 17, 68, 30, 57, 29, 58, 28, 59, 27, 60, 26, 61, 25, 62]
7	C_{28}	[7, 55, 33, 54, 21, 59, 15, 62, 27, 61, 16, 57, 20, 56, 17, 60, 18, 58, 26, 49, 25, 50, 24, 51, 23, 52, 22, 53]
6	C_{24}	[6, 46, 28, 45, 18, 50, 15, 49, 13, 52, 23, 51, 14, 47, 17, 48, 22, 41, 21, 42, 20, 43, 19, 44]
5	C_{20}	[5, 37, 23, 36, 15, 39, 14, 40, 12, 41, 19, 42, 11, 38, 18, 33, 17, 34, 16, 35]
4	C_{16}	[4, 28, 18, 27, 12, 30, 11, 32, 9, 29, 15, 31, 14, 25, 13, 26]
3	C_{12}	[3, 19, 11, 17, 8, 22, 9, 21, 10, 20, 13, 18]

Conditions : (3/4) p -1 < m ≤ p-3, m ≠ (5/6) p- (5/6)

Step	The successive vertex labels
1	(5p+4m-1, 3p, 5p+4m-2, 3p+1, ... , 4p-m-5, 4p+5m+3, 4p-m-4, 4p+5m+2)
2	(5p+4m+1, 5p-2m-4); (5p+4m, 7p-4m-5)
3	(4p+5m+1, 4p-m-3, 4p+5m, 4p-m-2, ... , 3p+6m+3, 5p-2m-5, 3p+6m+2, 5p-2m-4)
	Apply transformation type 2 to the vertex labels (5p-2m-3, 5p-2m-2, ... , 7p-4m-5, ... , 2p+2m, ... , 4p-3, 4p-2) and (4p+4m+1, 4p+4m+2, ... , 3p+6m, 3p+6m+1) by selecting the vertices 7p-4m-5 and 2p+2m as end vertices

Conditions : (3/4) p -1 < m ≤ p-3, m = (5/6) p- (5/6), m = 5 [Note: This case was discussed in case m = p+2]

Conditions : (3/4) p -1 < m ≤ p-3, m = (5/6) p- (5/6), m ≠ 5

Step	The successive vertex labels
1	(5p+4m-1, 3p, 5p+4m-2, 3p+1, ... , 4p-m-5, 4p+5m+3, 4p-m-4, 4p+5m+2)
2	(5p+4m+1,5p-2m-4); (5p+4m,7p-4m-6); (5p-2m-4 , 4p+4m+2); (2p+2m-1, 6p+2m- 2)
3	(4p+5m+1, 4p-m-3, 4p+5m, 4p-m-2, … , 3p+6m+3, 5p-2m-5, 3p+6m+2, 5p-2m-3, 3p+6m+1, 5p-2m-2, …, 2p+2m-3, 6p+2m+1, 2p+2m-2, 6p+2m, 2p+2m)
	Apply transformation type 3 to the vertex labels (6p+2m-1, 6p+2m-2, …. , 4p+4m+2, 4p+4m+1) and (4p-2, 4p-3, … , 2p+2m+2, 2p+2m+1) by considering the vertices 6p+2m-2 and 4p+4m+2 as end points

Conditions : m = (3/4) p-1, p > 4 [Note: For p = 4 this case was solved in case m = p+2]

Step	The successive vertex labels
1	(5p+4m-1, 3p, 5p+4m-2, 3p+1, ... , 4p-m-5, 4p+5m+3, 4p-m-4, 4p+5m+2)
2	(5p+4m,5p-2m-2); (5p+4m+1, p+4m+1); (2p+2m-1,6p+2m-3)
3	(4p+5m+1, 4p-m-3, 4p+5m, 4p-m-2, ... , 6p+2m, 2p+2m-2, 6p+2m-1, 2p+2m)
4	Apply transformation type 1 to the vertex labels (2p+2m+1, 2p+2m+2, ... , 4p-4, 4p-3, 4p-2) and (4p+4m+1, 4p+4m+2, ... , 6p+2m-3, 6p+2m-2) by considering the vertices 4p-3 and 6p+2m-3 as end vertices

Conditions : (2/3) p -1 ≤ m < (3/4) p –1, m ≠ (2/3) p -1

Step	The successive vertex labels
1	(5p+4m-1, 3p, 5p+4m-2, 3p+1, ... , 4p-m-5, 4p+5m+3, 4p-m-4, 4p+5m+2)
2	(5p-2m-4,5p+4m+1); (p+4m+2,5p+4m); (2p+2m,10m+6)
3	(4p+5m+1, 4p-m-3, 4p+5m, 4p-m-2, ... , 6p+2m-1, 2p+2m-1, 6p+2m-2, 2p+2m+1, ... , 3p+6m+4, 5p-2m-5, 3p+6m+3, 5p-2m-4)
4	Apply transformation type 1 to the vertex labels (5p-2m-3, 5p-2m-2, ... , p+4m+2, ... , 4p-3, 4p-2) and (4p+4m+1, 4p+4m+2, ... , 10m+6, ... , 3p+6m+1, 3p+6m+2) by using the two vertices p+4m+2 and 10m+6 as end vertices

Conditions : (2/3) p -1 ≤ m < (3/4) p –1, m = (2/3) p -1

Step	The successive vertex labels
1	(5p+4m-1, 3p, 5p+4m-2, 3p+1, ... , 4p-m-5, 4p+5m+3, 4p-m-4, 4p+5m+2)
2	(2p+2m, 4p+5m+1); (p+4m+2 ,5p+4m+1); (p+4m+3,5p+4m)
3	Apply transformation type 2 to the vertex labels (2p+2m+1, 2p+2m+2, 2p+2m+3, ... , p+4m+2, p+4m+3, ... , 4p-3, 4p-2) and (4p+4m+1, 4p+4m+2, ... , 4p+5m-1, 4p+5m) by choosing the two vertices p+4m+2 and p+4m+3 as end vertices

Condition : (3/5) p - 1 < m < (2/3) p -1

Step	The successive vertex labels
1	(5p+4m-1, 3p, 5p+4m-2, 3p+1, ... , 2p+2m-2, 6p+2m, 2p+2m-1, 6p+2m-1, 2p+2m+1, ... , 4p-m-4, 4p+5m+3, 4p-m-3, 4p+5m+2)
2	(p+4m+2, 5p+4m+1); (5p-2m-4, 5p+4m); (8p-2m-6, 2p+2m)

	(4p+5m+1, 4p-m-2, 4p+5m, 4p-m-1, ... , 7p-2, p+4m+1, 7p-3, p+4m+2)
3	Apply transformation type 1 to the vertex labels (p+4m+3, p+4m+4, ... , 5p-2m-4, ... , 4p-4, 4p-3, 4p-2) and (4p+4m+1, 4p+4m+2, ... , 8p-2m-6, ... , 7p-6, 7p-5, 7p-4) by considering the two vertices 5p-2m-4 and 8p-2m-6 as end vertices

Condition : $m = (3/5)p - 1$

Step	The successive vertex labels
1	(5p+4m-1, 3p, 5p+4m-2, 3p+1, 5p+4m-3, 3p+2, ... , 6p+2m, 2p+2m-1, 6p+2m-1, 2p+2m)
2	(2p+2m+1, 6p+2m-2, 2p+2m+2, 6p+2m-3, ... , 4p-m-4, 4p+5m+3, 4p-m-3, 4p+5m+2)
	(5p+4m+1, p+4m+3); (2p+8m+5, 2p+2m+1); (4p+5m+1, p+4m+3); (5p+4m, 5p-2m-2)
	Apply transformation type 1 to the vertex labels (p+4m+4, p+4m+5, ... , 5p-2m-2, ... , 4p-3, 4p-2) and (4p+4m+1, 4p+4m+2, ... , 2p+8m+5, ... , 4p+5m-1, 4p+5m) by selecting the two end points 5p-2m-2 and 2p+8m+5.

Conditions : $(4/7)p-(8/7) < m < (3/5)p-1$

Step	The successive vertex labels
1	(5p+4m-1, 3p, 5p+4m-2, 3p+1, 5p+4m-3, 3p+2, ... , 6p+2m, 2p+2m-1, 6p+2m-1, 2p+2m)
2	(p+4m+3, 5p+4m+1); (5p-3m-5, 5p+3m); (5p-2m-4,5p+4m); (2p+3m+1,4p+5m+2)
3	Apply transformation type 2 to the vertex labels (2p+2m+1, 2p+2m+2, ... , 5p-3m-5, ... , p+4m+3, ... , 4p-m-4, 4p-m-3) and (4p+5m+3, 4p+5m+4, ... , 6p+2m-3, 6p+2m-2) by selecting the two vertices 5p-3m-5 and p+4m+3 as end points.
4	(4p+5m+1, 4p-m-2, 4p+5m, 4p-m-1, ... , 5p+3m+2, 3p+m-3, 5p+3m+1, 3p+m-2, 5p+3m)
5	Apply transformation type 2 to the vertex labels (3p+m-1, 3p+m, 3p+m+1, ... , 2p+3m+1, ... , 5p-2m-4, ... , 4p-3, 4p-2) and (2p+4m+1, 2p+4m+2, ... , 5p+3m-2, 5p+3m-1) by using the two vertices 2p+3m+1 and 5p-2m-4 as end vertices

Conditions : $m = (4/7)p - (8/7)$

Step	The successive vertex labels
1	(5p+4m-1, 3p, 5p+4m-2, 3p+1, 5p+4m-3, 3p+2, ... , 6p+2m, 2p+2m-1, 6p+2m-1, 2p+2m)
2	(p+4m+3, 5p+4m+1); (2p+2m+1, 8p-2m-6); (5p-2m-4, 5p+4m); (p+4m+4, p+10m+6)
3	(2p+2m+1, 6p+2m-2, 2p+2m+2, 6p+2m-3, ... , p+4m+1, 7p-2, p+4m+2, 7p-3, p+4m+3)
4	(p+4m+4, 7p-4, p+4m+5, 7p-5, ... , 4p-m-3, 4p+5m+3, 4p-m-2, 4p+5m+1)
5	Apply transformation type 1 to the vertex labels (4p-m-1, 4p-m, 4p-m+1, ... , 5p-2m-4, ... , 4p-3, 4p-2) and (4p+4m+1, 4p+4m+2, ... , p+10m+6, ... , 4p+5m-1, 4p+5m) by considering the two vertices 5p-2m-4 and p+10m+6 as end vertices

Conditions : $(1/2)p < m < (4/7)p-(8/7)$

[Note: The rest of the edge labels are generated by the same method as the above case]

Step	The successive vertex labels
1	Apply transformation type 2 to the vertex labels(2p+2m+1, 2p+2m+2, ... , p+4m+3, ... , 5p-3m-5, ... , 4p-m-4, 4p-m-3) and (4p+5m+3, 4p+5m+4, ... , 6p+2m-3, 6p+2m-2) by selecting the two vertices 5p-3m-5 and p+4m+3 as end points.