The Q Method for the Second-Order Cone Programming

F. Alizadeh
RUTCOR, Rutgers University,
Piscataway, NJ 08554-8003, USA

Yu Xia
Graduate School of Management
Rutgers University
Newark NJ USA

Key words: Second order cone programming, Semidefinite programming, Interior point methods, Jordan algebra

Abstract

In this paper we present a new algorithm for solving the second order cone programming problems which we call the Q method. This algorithm is an extension of the Q method of Alizadeh Haeberly and Overton for the semidefinite programming problem.

1 Introduction

The second order cone programming (SOCP for short) is the following convex optimization problem:

$$\begin{array}{ll} \min & \sum_i \mathbf{c_i}^T \mathbf{x_i} \\ \text{s.t.} & \sum A_i \mathbf{x_i} = \mathbf{b} \\ & \mathbf{x_i} \succcurlyeq_{\mathcal{Q}} \mathbf{0} \quad i = 1, \dots, n \end{array} \tag{1}$$

Here each $\mathbf{x_i} \in \mathbb{R}^{n_i+1}$ and is indexed from 0 with the 0^{th} entry playing a special role. The notation $\mathbf{x} \succcurlyeq_{\mathcal{Q}} \mathbf{0}$ means that $\mathbf{x}$ is in the second order cone $\mathcal{Q}$ which consists of vectors $\mathbf{x}$ where $x_0 \geq \|\bar{\mathbf{x}}\|$, with $\bar{\mathbf{x}} = (x_1, \dots, x_n)^T$. This problem is more general than linear programming–for instance if each $\mathbf{x_i} \in \mathbb{R}^2$ then we get an LP. It contains convex quadratically constrained quadratic programs as a special case. It is also a special case of the semidefinite programming since $\mathbf{x} \in \mathcal{Q}$ means that the arrow-shaped matrix

$$\operatorname{Arw}(\mathbf{x}) = \begin{pmatrix} x_0 & \bar{\mathbf{x}}^T \\ \bar{\mathbf{x}} & x_0 I \end{pmatrix}$$

is positive semidefinite.

The SOCP problem has been surveyed and studied in a number of works before. We only mention a few here. In paper [4] duality theory, complementarity conditions and non-degeneracy conditions are studied specially in comparison to LP and SDP. In [9] Karmarkar's original algorithm is extended in a word by word manner from LP to SOCP. In [7] numerous applications, especially in engineering are surveyed. Finally in the recent survey [1] both the theory and applications of SOCP along with methods of transforming seemingly unrelated problems to SOCP are studied.

In this abstract we sketch a new algorithm which is extension of the so-called Q method for SDP. The Q method for SDP is defined and analyzed in [2, 3].

2 Algebraic and duality properties

In this section we establish some useful algebraic properties that are fundamental tools for analyzing SOCP problems. These are more thoroughly discussed in [1] and are based on Jordan algebraic techniques, see [6].

2.1 algebraic notions

First define a product $\circ : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$:

$$\mathbf{x} \circ \mathbf{z} = \left(\mathbf{x}^T\mathbf{z}, x_0z_1 + x_1z_0, \ldots, x_0z_n + x_nz_0\right)^T$$

Note that $\mathbf{x} \circ \mathbf{z} = \operatorname{Arw}(\mathbf{x})\,\mathbf{z}$. Then $(\mathbb{R}^{n+1}, \circ)$ forms an algebra where $\circ$ is commutative, but not associative. However, $\mathbf{x} \circ (\mathbf{x}^2 \circ \mathbf{z}) = \mathbf{x}^2 \circ (\mathbf{x} \circ \mathbf{z})$ holds. From this it follows that the subalgebra generated by a single elements $\mathbf{x}$ is associative. That is we can write $\mathbf{x}^p$ without regard to order of multiplication.

It is easily verified that the vector $\mathbf{e} = (1, 0, \ldots, 0)^T$ is the identity element. Furthermore, each $\mathbf{x}$ satisfies

$$\mathbf{x}^2 - 2x_0\mathbf{x} + (x_0^2 - \|\bar{\mathbf{x}}\|^2)\mathbf{e} = \mathbf{0}$$

The polynomial $t^2 - 2x_0t + (x_0^2 - \|\bar{\mathbf{x}}\|^2)$ is *the characteristic polynomial* of $\mathbf{x}$; it roots $\lambda_{1,2} = x_0 \pm \|\bar{\mathbf{x}}\|$ are its *eigenvalues.* $\operatorname{tr}\mathbf{x} = \lambda_1 + \lambda_2 = 2x_0$ is the trace and $\det(\mathbf{x}) = \lambda_1\lambda_2 = x_0^2 - \|\bar{\mathbf{x}}\|^2$ is the determinant.

The cone $\mathcal{Q}$ defined above can be shown to coincide with the set of $\mathbf{x}^2$ of $\mathbb{R}^{n+1}$. Every vector $\mathbf{x}$ can be written in the form

$$\mathbf{x} = \lambda_1\mathbf{c_1} + \lambda_2\mathbf{c_2}$$

where $\mathbf{c}_{1,2} = \frac{1}{2}(1, \pm\frac{\bar{\mathbf{x}}}{\|\bar{\mathbf{x}}\|})^T$. This is the counterpart of spectral decomposition in the algebra of symmetric matrices. The vectors $\mathbf{c}_{1,2}$ have a number of interesting properties. First they are idempotentns, $\mathbf{c_i}^2 = \mathbf{c_i}$. Second, their sum $\mathbf{c_1} + \mathbf{c_2} = \mathbf{e}$. Such pairs of vectors are called *Jordan frames.* Thus every vector in $\mathbb{R}^{n+1}$ can be written as linear combination of vectors of a Jordan frame, so for each vector $\mathbf{x}$ its Jordan frame plays much the same role as eigenvectors $\mathbf{q}$ play for real symmetric matrices (more precisely they play the role of rank one matrices $\mathbf{q}\mathbf{q}^T$, since every symmetric matrix can be written as $\sum_i \lambda_i\mathbf{q_i}\mathbf{q_i}^T$ with $(\lambda_i, \mathbf{q_i})$ an eigenvelue/eigenvector pair.)

We say two vectors $\mathbf{x}$ and $\mathbf{y}$ *operator commute* if and only if they share the same Jordan frame. In that case the matrices $\operatorname{Arw}(\mathbf{x})$ and $\operatorname{Arw}(\mathbf{y})$ commute.

There is another way of extending spectral decomposition of symmetric matrices to its "$\mathcal{Q}$" analog. In the algebra of symmetric matrices, if $X = Q^T\Lambda Q$ with Λ a diagonal matrix containing the eigenvalues of X, and Q an orthogonal matrix whose columns are the corresponding eigenvectors, then we may write $\Lambda = \lambda_1E_1 + \cdots + \lambda_nE_n$, where E_i is the matrix with its i, i entry one and everywhere else zero. The E_i form a Jordan frame in the algebra of symmetric matrices under $X \circ Y = (XY + YX)/2$ operation: each is an idempotent and they add up to the identity matrix; so is the set $\mathbf{q_i}\mathbf{q_i}^T$ for each set of orthonormal vectors $\mathbf{q_i}$. Thus, calling the Jordan frame E_i *the standard frame* we see that the spectral decomposition means that for any matrix X there is a rotation operator $\rho_Q(X) = Q^TXQ$ that maps the Jordan frame of X to the standard frame and preserves its eigenvalues. We can extend this fact to $(\mathbb{R}^{n+1}, \circ)$ algebra. First we need to fix a standard frame. Let $\mathbf{d}_{1,2} = \frac{1}{2}(1, \pm 1, \mathbf{0})^T$ be defined as the standard frame. Then the Jordan frame of any vector $\mathbf{x} \in \mathbb{R}^{n+1}$ can be turned by an orthogonal transformation into the standard frame, while preserving the eigenvalues. More precisely, let $\mathbf{x} = \lambda_1\mathbf{c_1} + \lambda_2\mathbf{c_2}$. Then

(2) $$Q_{\mathbf{x}}\mathbf{x} = \lambda_1\mathbf{d_1} + \lambda_2\mathbf{d_2},$$

where the matrix $Q_{\mathbf{x}}$ is defined as

$$Q_{\mathbf{x}} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & \mathbf{0}^T \\ 0 & \frac{x_1}{\|\bar{\mathbf{x}}\|} & -\frac{\bar{\bar{\mathbf{x}}}^T}{\|\bar{\mathbf{x}}\|} \\ \mathbf{0} & \frac{\bar{\bar{\mathbf{x}}}}{\|\bar{\mathbf{x}}\|} & I - \frac{\bar{\bar{\mathbf{x}}}\bar{\bar{\mathbf{x}}}^T}{\|\bar{\mathbf{x}}\|(\|\bar{\mathbf{x}}\| + x_1)} \end{pmatrix},$$

$\bar{\bar{\mathbf{x}}} = (x_2, \ldots, x_n)^T$, and $\mathbf{c}_{1,2} \neq \mathbf{d}_{1,2}$, (otherwise $Q_{\mathbf{x}}$ is simply the identity matrix.) If $\mathbf{x}$ and $\mathbf{y}$ operator commute, they share a Jordan frame and thus $Q_{\mathbf{x}} = Q_{\mathbf{y}}$.

In symmetric matrices, it is well-known that if S is skew-symmetric, then the matrix $\exp(S) = \sum_{k=0}^{\infty} \frac{S^k}{k!}$ is orthogonal. In SOCP we are interested only in orthogonal matrices of the form $Q_{\mathbf{x}}$. It turns out that only a subset of skew-symmetric matrices generates all $Q_{\mathbf{x}}$ via the matrix exponential function. More precisely, let L be the set of all $Q_{\mathbf{x}}$'s, i.e. the set of matrices of the form

$$\begin{pmatrix} 1 & 0 & \mathbf{0}^T \\ 0 & c_0 & -\bar{\mathbf{c}}^T \\ \mathbf{0} & \bar{\mathbf{c}} & I - \frac{\bar{\mathbf{c}}\bar{\mathbf{c}}^T}{1+c_0} \end{pmatrix} : \mathbf{c} \in \mathbb{R}^{n+1},\ \|\mathbf{c}\| = 1,\ c_0 \neq -1$$

plus the additional matrix

$$\begin{pmatrix} 1 & 0 & 0 & \mathbf{0}^T \\ 0 & -1 & 0 & \mathbf{0}^T \\ 0 & 0 & -1 & \mathbf{0}^T \\ \mathbf{0} & 0 & 0 & I \end{pmatrix}$$

The decomposition (2) is unique if we restrict the orthogonal matrix to the set L. Define also the set $\mathfrak{l}$

$$\mathfrak{l} \stackrel{\text{def}}{=} \left\{ S_\mathbf{s} = \begin{pmatrix} 0 & 0 & \mathbf{0}^T \\ 0 & 0 & \mathbf{s}^T \\ \mathbf{0} & -\mathbf{s} & 0 \end{pmatrix} : \|\mathbf{s}\| \le \pi \right\}.$$

Then we have

PROPOSITION 1 *There is a one-to-one correspondence between L and $\mathfrak{l}$ through the exponential map* $\exp(\cdot)$.

Proof: For any $S_\mathbf{s} \in \mathfrak{l}$,

$$S_\mathbf{s}^2 = \begin{pmatrix} 0 & 0 & \mathbf{0}^T \\ 0 & -\mathbf{s}^T\mathbf{s} & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & -\mathbf{s}\mathbf{s}^T \end{pmatrix}, \tag{3}$$

and

$$S_\mathbf{s}^{(2k+1)} = (-\mathbf{s}^T\mathbf{s})^k S_\mathbf{s}, \quad S_\mathbf{s}^{(2k+2)} = (-\mathbf{s}^T\mathbf{s})^k S_\mathbf{s}^2.$$

Hence for $\|\mathbf{s}\| \neq 0$,

$$\begin{aligned} \exp(S_\mathbf{s}) &= I + \frac{S_\mathbf{s}^2}{\|\mathbf{s}\|^2} \left[\sum_{i=1}^{\infty} (-1)^{i+1} \frac{\|\mathbf{s}\|^{2i}}{(2i)!} \right] \\ &\quad + \frac{S_\mathbf{s}}{\|\mathbf{s}\|} \left[\sum_{i=0}^{\infty} (-1)^{i} \frac{\|\mathbf{s}\|^{2i+1}}{(2i+1)!} \right] \\ &= I + \frac{1-\cos(\|\mathbf{s}\|)}{\|\mathbf{s}\|^2} S_\mathbf{s}^2 + \frac{\sin(\|\mathbf{s}\|)}{\|\mathbf{s}\|} S_\mathbf{s}. \end{aligned}$$

We use $Q_\mathbf{c}$ to emphasize the dependence of $Q \in L$ on ($\mathbf{c} \in \mathbb{R}n+1$ with $\|\mathbf{c}\| = 1$) in this proof.

Notice $\exp(0) = I = Q_\mathbf{e}$ and for $\mathbf{s} = \pi\mathbf{e}$,

$$\exp(S_\mathbf{s}) = \begin{pmatrix} 1 & 0 & 0 & \mathbf{0}^T \\ 0 & -1 & 0 & \mathbf{0}^T \\ 0 & 0 & -1 & \mathbf{0}^T \\ \mathbf{0} & 0 & 0 & I \end{pmatrix} = Q_{-\mathbf{e}}.$$

For any $\mathbf{c} \in \mathbb{R}^{n+1}$, $\|\mathbf{c}\| = 1$, $|c_0| \neq 1$, there is a unique $0 < \alpha < \pi$ such that $\cos\alpha = c_0$ and $\sin\alpha = \|\bar{\mathbf{c}}\|$. Since $\|\bar{\mathbf{c}}\| \neq 0$, let $\mathbf{s} = -\frac{\alpha}{\|\bar{\mathbf{c}}\|}\bar{\mathbf{c}}$, then $\exp(S_\mathbf{s}) = Q_\mathbf{c}$.

On the other hand, $\forall S_\mathbf{s} \in \mathfrak{l}, S_\mathbf{s} \neq 0$, let $\bar{\mathbf{c}} = -\frac{\sin(\|\mathbf{s}\|)}{\|\mathbf{s}\|}\mathbf{s}$, $c_0 = \cos\|\mathbf{s}\|$, then $\exp(S_\mathbf{s}) = Q_\mathbf{c} \in L$. ■

COROLLARY 1 *For any $Q_{\mathbf{c}_1} \in L$, $Q_{\mathbf{c}_2} \in L$, we have*
(1) $Q_{\mathbf{c}_1}Q_{\mathbf{c}_2} \in L$, and
(2) $\exists Q_{\mathbf{c}_3} \in L$ such that $Q_{\mathbf{c}_2} = Q_{\mathbf{c}_1}Q_{\mathbf{c}_3}$. Thus the set L is a subgroup of the orthogonal group.

Proof: By Proposition 1, $\exists \mathbf{s}_1 \in \mathfrak{l}$, $\mathbf{s}_2 \in \mathfrak{l}$, such that $Q_{\mathbf{c}_1} = \exp(S_{\mathbf{s}_1})$, $Q_{\mathbf{c}_2} = \exp(S_{\mathbf{s}_2})$. We ignore the constraints $\|\mathbf{s}\| \le \pi$, the above proof is still effective except the uniqueness. Hence $Q_{\mathbf{c}_1}Q_{\mathbf{c}_2} = \exp(S_1 + S_2) \in L$ and $Q_{\mathbf{c}_3} = \exp(S_2 - S_1) \in L$. ■

COROLLARY 2 *For any Q in L, $Q(\mathcal{Q}) = \mathcal{Q}$.*

Proof: Omitted. ■

2.2 Geometric notion

We now discuss the duality and complementarity conditions of the SOCP problem. As in LP and SDP there is a dual associated with (1) problem

$$\begin{array}{ll} \max & \mathbf{b}^T\mathbf{y} \\ \text{s.t.} & A_i\mathbf{y} + \mathbf{z_i} = \mathbf{c_i} \quad i = 1, \ldots, n \\ & \mathbf{z_i} \succcurlyeq_\mathcal{Q} \mathbf{0} \end{array} \tag{4}$$

It can be shown that if both the primal problem (1) and the dual problem (4) have feasible solutions in the interior of the cone $\mathcal{Q}$, then the the optimal values of the primal and dual problems coincide. At the optimum the complementary slackness condition holds which can be expressed succinctly as $\mathbf{x_i} \circ \mathbf{z_i} = \mathbf{0}$ for $i = 1, \ldots, n$.

3 The Q method

The Q method to be discussed below is a variation of primal-dual interior point methods for SOCP (see for example [4, 1]). The basic strategy of such algorithms is to apply the Newton's method to primal feasibility, dual feasibility (that is the equality constraints in (1) and (4)), and the following form of relaxed complementarity: $\mathbf{x_i z_i} = \mu \mathbf{e}$, or some appropriate transformation of them. The set of $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ that satisfy these equations traverses a path by varying μ; this path is called the *central path*. As $\mu \to 0$ $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ approaches the primal and dual optimum solution. Each iteration involves one application of Newton's method along with a reduction of μ by a constant factor.

The idea behind the Q method is to note that at the central path $\mathbf{x_i}$ and $\mathbf{z_i}$ operator commute, and thus share a common Jordan frame. Thus instead of $\mathbf{x_i}$ and $\mathbf{z_i}$ we deal with $\boldsymbol{\lambda}$—the vector of eigenvelues of all $\mathbf{x_i}$ and $\boldsymbol{\omega}$—the vector of eigenvalues of all $\mathbf{z_i}$. Thus, on the central path, each iterate $(\boldsymbol{\lambda}, \boldsymbol{\omega}, \mathbf{y}, Q)$ satisfies

$$
\begin{aligned}
Q\tilde{P}\boldsymbol{\omega} + A^T\mathbf{y} &= \mathbf{c}, \\
AQ\tilde{P}\boldsymbol{\lambda} &= \mathbf{b}, \\
\Lambda\Omega &= \mu I,
\end{aligned}
\tag{5}
$$

where $\tilde{P}$ is block diagonal, its i^{th} block, denoted as $\tilde{P}_i \in \mathbb{R}^{(n_i+1)\times 2}$, is in the form

$$
\tilde{P}_i \stackrel{\text{def}}{=} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \\ \vdots & \vdots \end{pmatrix}.
$$

Since the decomposition (2) is unique when $Q \in L$, if (1) has an interior feasible solution and A has full row rank, then $\forall\, \mu > 0$, (5) has a unique solution $(\boldsymbol{\lambda}_\mu, \boldsymbol{\omega}_\mu, \mathbf{y}_\mu, Q_\mu)$, and as $\mu \to 0$, $(\boldsymbol{\lambda}_\mu, \boldsymbol{\omega}_\mu, \mathbf{y}_\mu, Q_\mu)$ tends to the optimum of (1). Denote each diagonal block of Q by Q_i, by Proposition 1, we can approximate $Q_i \in L$ by $Q_i \exp(S_i)$ with $S_i \in \mathfrak{l}$ and then discard the nonlinear terms of the power expansion of $\exp(S_i)$. Define

$$
\mathbf{r}_p \stackrel{\text{def}}{=} \mathbf{b} - A\mathbf{x}, \quad \mathbf{r}_d \stackrel{\text{def}}{=} \mathbf{c} - \mathbf{z} - A^T\mathbf{y}, \quad \mathbf{r}_c \stackrel{\text{def}}{=} \operatorname{vec}(\mu I - \Lambda\Omega).
$$

Given an iterate $(\mathbf{x}^k, \mathbf{y}^k, \mathbf{z}^k) = \left(Q^k\tilde{P}\boldsymbol{\lambda}^k, \mathbf{y}^k, Q^k\tilde{P}\boldsymbol{\omega}^k\right)$, we denote $B^k \stackrel{\text{def}}{=} AQ^k$. The system of equations that defines the search direction is the following.

$$
\begin{aligned}
\tilde{P}\Delta\boldsymbol{\omega} + SP\boldsymbol{\omega}_k + {B^k}^T\Delta\mathbf{y} &= {Q^k}^T\mathbf{r}_d^k, \\
B^k\tilde{P}\Delta\boldsymbol{\lambda} + B^kS\tilde{P}\boldsymbol{\lambda}^k &= \mathbf{r}_p^k, \\
\Lambda^k\Delta\boldsymbol{\omega} + \Omega^k\Delta\boldsymbol{\lambda} &= \mathbf{r}_c^k.
\end{aligned}
\tag{6}
$$

Let P be a block diagonal matrix with each diagonal block in the form $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$. It is obvious that $P^{-1} = 2P$. After collecting all the first two columns of B_i^k into $\bar{B}^k$, the remaining columns into $\hat{B}^k$, and splitting ${Q^k}^T\mathbf{r}_d^k$ accordingly as $\bar{\mathbf{r}}_d^k$ and $\hat{\mathbf{r}}_d^k$, we rewrite the Newton system as

$$
\begin{aligned}
P\Delta\boldsymbol{\omega} + (\bar{B}^k)^T\Delta\mathbf{y} &= \bar{\mathbf{r}}_d^k, \\
\frac{(\omega_i)_2 - (\omega_i)_1}{2}\mathbf{s}_i + (\hat{B}_i^k)^T\Delta\mathbf{y} &= (\hat{\mathbf{r}}_d)_i^k \quad (i = 1, \ldots, n), \\
\bar{B}^kP\Delta\boldsymbol{\lambda} + \sum_{i=1}^{n}\left[\frac{(\lambda_i)_2 - (\lambda_i)_1}{2}\hat{B}_i^k\mathbf{s}_i\right] &= \mathbf{r}_p^k, \\
\Lambda^k\Delta\boldsymbol{\omega} + \Omega^k\Delta\boldsymbol{\lambda} &= \mathbf{r}_c^k.
\end{aligned}
\tag{7}
$$

Define $E_i \stackrel{\text{def}}{=} \frac{(\boldsymbol{\omega}_i)_2 - (\boldsymbol{\omega}_i)_1}{2}I$, $D_i \stackrel{\text{def}}{=} \frac{(\boldsymbol{\lambda}_i)_2 - (\boldsymbol{\lambda}_i)_1}{2}I$, where $I \in \mathbb{R}^{(n_i-2)\times(n_i-2)}$ is the identity matrix. Accordingly, define $E \stackrel{\text{def}}{=} \operatorname{Diag}(E_i)$, $D \stackrel{\text{def}}{=} \operatorname{Diag}(D_i)$. Hence, solution to (7) is

$$
\begin{aligned}
\Delta\mathbf{y} = {} & \left(\bar{B}P2\Omega^{-1}\Lambda P^T\bar{B}^T - \hat{B}DE^{-1}\hat{B}^T\right)^{-1} \times \\
& \Big(\mathbf{r}_p - \bar{B}P\Omega^{-1}\mathbf{r}_c - \hat{B}DE^{-1}\hat{\mathbf{r}}_d \\
& + \bar{B}P2\Omega^{-1}\Lambda P^T\bar{\mathbf{r}}_d\Big) \\
\Delta\boldsymbol{\omega} = {} & P^{-1}\left(\bar{\mathbf{r}}_d - \bar{B}^T\Delta\mathbf{y}\right) \\
\Delta\boldsymbol{\lambda} = {} & \Omega^{-1}\left(\mathbf{r}_c - \Lambda\Delta\boldsymbol{\omega}\right) \\
\mathbf{s} = {} & 2E^{-1}\left(\hat{\mathbf{r}}_d - \hat{B}^T\Delta\mathbf{y}\right)
\end{aligned}
\tag{8}
$$

The inverse involved in (8) is applied to a symmetric positive semidefinite matrix, so we can use Cholesky factorization to compute the search direction.

Now we can state the basic algorithm of Q method for SOCP.

Given $\epsilon_p > 0, \epsilon_d > 0, \epsilon_c > 0$, we first choose $0 \leq \sigma < 1$. Then, the basic algorithm for the Q method is

Do until $\|\mathbf{r}_p^k\| \le \epsilon_p$, $\|\mathbf{r}_d^k\| \le \epsilon_d$, and ${\boldsymbol{\lambda}^k}^T \boldsymbol{\omega}^k \le \epsilon_c$.

1. Set $\mu = \sigma \frac{{\boldsymbol{\lambda}^k}^T \boldsymbol{\omega}^k}{2n}$.

2. Compute the search direction $(\Delta\boldsymbol{\lambda}, \Delta\boldsymbol{\omega}, \Delta\mathbf{y}, \mathbf{s})$ from (7),

3. Choose step sizes α, β, γ, so $\boldsymbol{\lambda}^{k+1} > 0$, $\boldsymbol{\omega}^{k+1} > 0$; set

$$\begin{aligned}
\Lambda^{k+1} &= \Lambda^k + \alpha\Delta\Lambda, \\
\mathbf{y}^{k+1} &= \mathbf{y}^k + \beta\Delta\mathbf{y}, \\
\Omega^{k+1} &= \Omega^k + \beta\Delta\Omega, \\
Q^{k+1} &= Q^k \left(I + \frac{1}{2}\gamma S\right)\left(I - \frac{1}{2}\gamma S\right)^{-1};
\end{aligned}$$

4. $k \leftarrow k + 1$.

End

When the dimension of $\mathbf{x}_i$ is two, $\forall\, k \ge 1$, we set $Q_i^k = I$ and $S_i^k = 0$.

For the step sizes, simply, we choose $\alpha = \min(1, \tau\hat{\alpha})$, $\beta = \min(1, \tau\hat{\beta})$, $\gamma = \sqrt{\alpha\beta}$, where $\hat{\alpha}$ and $\hat{\beta}$ are primal and dual step sizes to the boundary of positive orthant, $0 < \tau < 1$.

To avoid calculation of sin and cos, in the above algorithm, we ignore the constraint ($Q_i^k \in L$) to ($Q_i^k \in K$), where

$$K \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} 1 & \\ & \bar{Q} \end{pmatrix} : \bar{Q} \text{ being real orthogonal} \right\},$$

and replace the update ($Q^{k+1} = Q^k \exp(\gamma S)$) by the Cayley transformation.

When $\left\|\frac{\alpha}{2}\mathbf{s}\right\| < 1$, according to Neumann Lemma, $(I - \frac{\alpha}{2}S)^{-1}$ can be expanded by power series. So the Cayley transformation is

$$(I + \frac{\alpha}{2}S)(I - \frac{\alpha}{2}S)^{-1} = I + \sum_{k=1}^{+\infty} \frac{\alpha^k}{2^{k-1}} S^k.$$

By (3), each block of the Cayley transformation is equivalent to

(9)
$$(I + \frac{\alpha}{2}S_i)(I - \frac{\alpha}{2}S_i)^{-1} = I + \alpha S_i - \frac{\alpha^3 \|\mathbf{s}_i\|^2}{4 + \alpha^2 \|\mathbf{s}_i\|^2} S_i + \frac{2\alpha^2}{4 + \alpha^2 \|\mathbf{s}_i\|^2} S_i^2.$$

Hence, we need not calculate inverse when applying Cayley transformation.

Cost of computation the search direction from (8) is less than that from (7), but if $(\boldsymbol{\omega}_i^k)_2 - (\boldsymbol{\omega}_i^k)_1$ is zero or almost zero, round-off errors may cause problem. In the implementation, when $((\boldsymbol{\omega}_i^k)_2 - (\boldsymbol{\omega}_i^k)_1)$ is not too small, we use (8); otherwise, we solve (7) directly.

The key results of our research are summarized in the following two statements. We omit the proofs for lack of space. Interested reader may refer to the more complete manuscript [5] for the details.

THEOREM 1 *For the iterates generated by the Q method,*

1. *When at the optimum both primal and dual solutions are non-degenerate and satisfy strict complementarity (see [4, 1] for definitions and characterizations) the Jacobian of the Newton system is non-singular at the optimum, thereby guaranteeing asymptotic Q order of convergence of two.*

2. *The Q method is is globally convergent if sufficiently small step length is taken at each iteration.*

We adopt the analysis presented in [8] for LP to prove global convergence of the Q method.

References

[1] F. Alizadeh and D. Goldfarb. Second-order cone programming. Submitted to Mathematical Programming, August 2001.

[2] F. Alizadeh, J.P. Haeberly, and M.L. Overton. A new primal-dual interior point method for semidefinite programming. In *Proc. 5th SIAM Conference on Appl. Linear Algebra*, Snowbird, Utah, 1994.

[3] F. Alizadeh, J.P. Haeberly, and M.L. Overton. Primal-dual interior-point methods for semidefinite programming: Convergence rates, stability and numerical results. *SIAM J. Optim.*, 8(3):746–768, 1998.

[4] F. Alizadeh and S.H. Schmieta. Optimization with Semidefinite, Quadratic and Linear Constraints. Technical Report rrr 23-97, RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway, NJ 08854-8803, November 1997. Available at URL: `http://rutcor.rutgers.edu/pub/rrr/reports97/23.ps`.

[5] F.Alizadeh and Y.Xia. The Q method for the Second Order Cone Programming. Manuscript, October 2001.

[6] J. Faraut and A. Korányi. *Analysis on Symmetric Cones.* Oxford University Press, Oxford, UK, 1994.

[7] M. S. Lobo, L. Vandenberghe, S. Boyd, and H. Lebret. Second order cone programming. *Linear Algebra Appl.*, 284:193–228, 1998.

[8] M.Kojima, N.Megiddo, and S.Mizuno. A primal-dual infeasible-interior-point algorithm for linear programming. *Math. Programming*, 61:263–280, 1993.

[9] A. Nemirovski and K. Scheinberg. Extension of Karmarkar's Algorithm onto convex quadratically constrained quadratic programming. *Math. Programming*, 72:273–289, 1996.