Regular Filter Space

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Abstract:- A necessary and sufficient condition is obtained for a filter space to have a regular completion. Also, T_3 and strict regular completions of a T_2 filter space are discussed as special cases. Subsequently, different completion functors and completion subcategories are constructed for the category FIL.

Key-Words:- Filter space, c-regular, c-separated, regular completion, strict completion.

1 Introduction

Filter spaces are generalizations of Cauchy spaces, which were studied by Kent, Richardson and several others [2,7,8] in the past few decades. Categorists [1,5] have shown that the category FIL of filter spaces forms a strong topological universe, since it is isomorphic to the filter-merotopic spaces. It is also cartesian closed and preserves the quotient structure. The category CHY of Cauchy spaces is a bireflective subcategory of FIL. A completion theory was developed and a completion functor was defined for FIL by the author and one of her co-authors [3]. As we have seen in [3], the Wyler completion of filter spaces has nice functorial properties, but it does not preserve regularity and uniformizability. In this paper, different completions on certain subcategories of FIL are constructed which preserve reularity and other important characteristics. The completion theory for regular filter spaces, developed here generalizes several of Kent and Richardson's techniques [2,4] for completions of regular Cauchy spaces, the study of which was initiated by Ramaley and Wyler [7]. In Section 2, some of the frequently used notations and definitions are presented together with the two basic problems discussed in later sections. In Section 3, a regularity series for filter spaces is defined for obtaining a regular modification, which later leads to the regular completion of these spaces. This section also generalizes the concept of regular completion of Cauchy spaces without the T_2 restriction, which was investigated by the author in [8]. Finally, in Section 4, strict regular completion of filter spaces are discussed. Some of the completions discussed in this paper also have the universal extension property, which leads to the definition of corresponding completion functors on subcategories of FIL. The author intends to carry on a more intensive investigation of completion functors and modification functors for FIL in a later paper.

2 Preliminaries

The following are some of the basic definitions and notations which will be frequently used throughout the paper. Let X be a nonempty set and $\mathbf{F}(X)$ be the set of all filters on X, partially ordered by inclusion. If $\mathcal{B}$ is a *base* [9] of the filter $\mathcal{F}$, then we write $\mathcal{F} = [\mathcal{B}]$ and $\mathcal{F}$ is said to be *generated* by $\mathcal{B}$. In particular, $\dot{x} = [\{x\}]$ and $\mathcal{F} \cap \mathcal{G} = [\{F \cup G \mid F \in \mathcal{F},\ G \in \mathcal{G}\}]$. If $F \cap G \neq \phi$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$, then $\mathcal{F} \vee \mathcal{G}$ =[{ F$\cap G \mid F \in \mathcal{F},\ G \in \mathcal{G}\}]$. If there exists $F \in \mathcal{F}$, $G \in \mathcal{G}$ such that $F \cap G = \phi$, then we say that $\mathcal{F} \vee \mathcal{G}$ *fails to exist.* If $A \subseteq X$, $\mathcal{F} \in \mathbf{F}(X)$, then $\mathcal{F}_A = [\{F \cap A \mid F \in \mathcal{F}\}]$ is the *trace* of $\mathcal{F}$ on A.

Definition 2.1 Let X be a set and $C \subseteq \mathbf{F}(X)$. The pair (X, C) is called a *filter space*, if the following conditions hold :

(c_1) $\dot{x} \in C$, $\forall x \in X$;

(c_2) $\mathcal{F} \in C$ and $\mathcal{G} \geq \mathcal{F}$ imply that $\mathcal{G} \in C$.

If (X, C) and (X, D) are two filter spaces and $C \subseteq D$, then C is said to be *finer* than D, written $C \geq D$.

For a set $\mathcal{A} \subseteq \mathbf{F}(X)$, two filters $\mathcal{F}, \mathcal{G} \in \mathcal{A}$ are said to be $\mathcal{A} - linked$, written $\mathcal{F} \sim_{\mathcal{A}} \mathcal{G}$, if there exist a finite number of filters $\mathcal{H}_1,\ \mathcal{H}_2, \cdots, \mathcal{H}_n$ in $\mathcal{A}$ such that $\mathcal{F} \vee \mathcal{H}_1,\ \mathcal{H}_1 \vee \mathcal{H}_2,\ \mathcal{H}_2 \vee \mathcal{H}_3, \cdots, \mathcal{H}_{n-1} \vee \mathcal{H}_n,\ \mathcal{H}_n \vee \mathcal{G}$ exist. In particular, if $\mathcal{A} = C$ then $\sim_c$ defines an equivalence relation on C. Let $[\mathcal{F}]_c$ denote the equivalence class containing $\mathcal{F} \in C$. Associated with the filter space (X, C), there is a preconvergence structure[3] p_c defined as

$$\mathcal{F} \overset{p_c}{\rightarrow} x \ \ if\ and\ only\ if\ \mathcal{F} \sim_c \dot{x}. \ (1)$$

A filter space (X, C) is a *c-filter space*, if $\mathcal{F} \sim_c \dot{x} \ \Rightarrow\ \mathcal{F} \cap \dot{x} \in C$ and it is a *Cauchy space*, if $\mathcal{F} \sim_c \mathcal{G} \ \Rightarrow\ \mathcal{F} \cap \mathcal{G} \in C$. It is known that the category $CFIL$ of c-filter spaces and the category CHY of Cauchy spaces are bireflective subcategories of FIL of filter spaces [1]. The preconvergence structure p_c associated with the filter space (X, C) is a convergence structure [6] iff (X, C) is a c-filter space.

For any $A \subseteq X$, we define $cl_{p_c} A = \{x \in X \mid \exists \mathcal{F} \in C$ such that $\mathcal{F} \overset{p_c}{\rightarrow} x$ and $A \in \mathcal{F}\}$ and $cl_{p_c} \mathcal{A} = [\{cl_{p_c} A \mid A \in \mathcal{A}\}]$. A filter space is said to be

- T_2 or *Hausdorff* iff $x = y$, whenever $\dot{x} \sim_c \dot{y}$.
- *regular* iff $cl_{p_c} \mathcal{F} \in C$, whenever $\mathcal{F} \in C$,
- *complete* iff each $\mathcal{F} \in C$ p_c converges.
- *locally compact* iff $\mathcal{F}$ p_c converges implies that $\exists\ F \in \mathcal{F}$ such that each ultrafilter containing F p_c converges to a point in F.

Note that a regular filter space is a c-filter space.

Let $A \subseteq X$ and $C_A = \{\mathcal{G} \in \mathbf{F}(A) \mid$ there exists $\mathcal{F} \in C$ such that $\mathcal{G} \geq \mathcal{F}_A\}$. Then (A, C_A) is a filter space, called the *subspace* of (X, C). A mapping $f : (X, C) \rightarrow (Y, D)$ is *continuous* if and only if $\mathcal{F} \in C$ implies that $f(\mathcal{F}) \in D$. The map f is a *homeomorphism* if and only if it is bijective and both f and f^{-1} are continuous. Moreover, f is an *embedding* of (X, C) into (Y, D) if and only if $f : (X, C) \rightarrow (f(X), D_{f(X)})$ is a homeomorphism.

A *completion* of a filter space (X, C) is a pair $((Z, K), \phi)$, where (Z, K), is a complete filter space and the map $\phi : (X, C) \rightarrow (Z, K)$ is an embedding satisfying the condition $cl_{p_k} \phi(X) = Z$. The completion $((Z, K), \phi)$ is said to be T_2 (respectively, regular, T_3), if the filter space $((Z, K)$ is T_2 (respectively, regular, T_3). A T_2 Wyler completion was constructed for filter spaces in [3]. Next, we generalize the concept of stable completion of Cauchy spaces [8] to filter spaces.

Definition 2.2 A completion $((Z, K), \phi)$ of a filter space (X, C) is said to be *stable* , if $z \in Z \setminus \phi(X)$ and $\phi(\mathcal{F}) \overset{p_k}{\rightarrow} z$ for some $\mathcal{F} \in C$ imply that z is the unique limit of $\phi(\mathcal{F})$ in Y.

The category REG of regular uniform convergence spaces is a bireflective subcategory of $SUCONV$ of semiuniform convergence spaces [5] Since $RFIL$, the category of regular filter spaces forms a subcategory of $CFIL$, which generalises semiuniform convergence spaces, the study of completion subcategories for $RFIL$ is important. In the next two sections attempts have been made to
(1) find characterization of filter spaces which have regular completion,
(2) construct strict completions under suitable conditions on the filter space.

3 Regular Completions

Let $\mathcal{A} \subseteq \mathbf{F}(X)$ and $\mathcal{A}' = \{\dot{x} \mid x \in X\} \cup \mathcal{A}$. We define the following sets :

- $PC_{\mathcal{A}} = \{\mathcal{F} \in \mathbf{F}(X) \mid \exists \mathcal{G} \in \mathcal{A}'$ with $\mathcal{F} \geq \mathcal{G}\}$.
- $q\text{-}C_{\mathcal{A}} = \{\mathcal{F} \in \mathbf{F}(X) \mid \exists \mathcal{G} \in \mathcal{A}'$ and a finite number of elements $x_1,\ x_2, \cdots, x_n \in X$ with $\mathcal{G} \sim_{\mathcal{A}'} x_i$ for each $i = 1, \cdots n$, such that $\mathcal{F} \geq \mathcal{G} \cap \dot{x_1} \cap \cdots \cap \dot{x_n}\}$.
- $CC_{\mathcal{A}} = \{\mathcal{F} \in \mathbf{F}(X) \mid \exists$ a finite number of $\mathcal{A}'$-linked filters $\mathcal{H}_1, \cdots, \mathcal{H}_n$ such that $\mathcal{F} \geq \cap_{i=1}^{n} \mathcal{H}_i\}$.

Proposition 3.1 The following statements are true for $PC_{\mathcal{A}}$, $q\text{-}C_{\mathcal{A}}$ and $CC_{\mathcal{A}}$:
(I) $PC_{\mathcal{A}}$ is the finest filter structure on X containing $\mathcal{A}$.
(II) $q\text{-}C_{\mathcal{A}}$ is the finest c-filter structure on X containing $\mathcal{A}$.
(III) $CC_{\mathcal{A}}$ is the finest Cauchy structure on X containing $\mathcal{A}$.

The proof of (I) is clear from the construction of $PC_{\mathcal{A}}$ and the proofs of (II) and (III) follow from Proposition 1.11 in [3], if we replace C by $\mathcal{A}$.

Next, following the idea in [2], we construct a regularity series for a filter space (X, C) which is used later in the section in costructing a regular completion.

Let $r_0 C = C$, $r_1 C = PC_{\mathcal{A}_1}$, where $\mathcal{A}_1 = \{cl^n_{p_{r_0 C}} \mathcal{F} \mid \mathcal{F} \in C,\ n \in N\}$ and $r_2(C) = PC_{\mathcal{A}_2}$, where $\mathcal{A}_2 = \{cl^n_{p_{r_1 C}} \mathcal{F} \mid \mathcal{F} \in C,\ n \in N\}$. In general, $r_\beta(C) = PC_{\mathcal{A}_\beta}$, where $\mathcal{A}_\beta = \{cl^n_{p_{r_{\beta-1}(c)}} \mathcal{F} \mid \mathcal{F} \in C,\ n \in N\}$, if β is a non-limit ordinal and $r_\beta(C) = \cup\{r_\alpha(C) \mid \alpha \leq \beta\}$, β is a limit ordinal. From the construction it is clear that $C = r_0 C \geq r_1 C \geq \cdots \geq r_\beta(C) \geq r_{\beta+1}(C)$, for all ordinal

numbers β.
The *length* l_r of a regularity series r for a filter space (X, C) is the smallest ordinal number γ for which $r_\gamma(C) = r_{\gamma+1}(C)$. Let rC, called the regular modification of C, be the finest regular filter structure on X which is coarser than C.

Proposition 3.2 (I) $C \geq r_\beta(C) \geq r_\gamma(C) \geq rC$, for any any ordinal numbers $\gamma \geq \beta$.
(II) $r_\gamma(C) = rC$ iff $\gamma \geq l_r$.

Proof. The proof of (I) follows by transfinite induction. To prove (II), first note that $rC \leq r_\gamma(C)$ follows from (I). Let $\gamma \geq l_r$. If γ is a limit ordinal, then $\mathcal{F} \notin r_\gamma(C) \Rightarrow \mathcal{F} \notin r_\alpha(C),\ \forall\ \alpha < \gamma$. Since each $r_\alpha(C)$ is regular and coarser than C, it follows that $\mathcal{F} \notin rC$. Next let γ be a non-limit ordinal. So $\mathcal{F} \notin r_\gamma(C) \Rightarrow \exists$ a regular filter structure $r_\gamma(C)$ on X which is coarser than C such that $\mathcal{F} \notin r_\gamma(C)$. Hence $\mathcal{F} \notin rC$. So $rC \geq r_\gamma(C)$. Conversely, let $r_\gamma(C) = rC$. If possible, let $\gamma < l_r$. So $r_\gamma(C) \neq r_{\gamma+1}(C)$. Hence $\exists\ \mathcal{F} \in r_{\gamma+1}(C)$ such that $\mathcal{F} \notin r_\gamma(C) = rC$, which implies that $\mathcal{F} \notin r_{\gamma+1}(C)$ (by (I)). This leads to a contradiction. Therefore, $\gamma \geq l_r$. This proves Proposition 3.2 . $\diamond$

Proposition 3.3 If (X, C) is a complete filter space, then $(X, r_\beta(C))$ is complete for each ordinal number β.

Proof. $(X, r_0C)) = (X, C)$ is complete. Next, let $\mathcal{G} \in r_1C$. So $\mathcal{G} \geq cl^n_{p_{r_0C}}\mathcal{F}$ for some $n \in N$ and $\mathcal{F} \in C$. Since (X, C) is complete $\exists\ x \in X$ such that $\mathcal{F} \sim_c \dot{x}$. $C \geq r_1C$ and $cl_{p_{r_0C}}\mathcal{F} \in r_1C \Rightarrow \mathcal{G} \sim_{r_1C} \dot{x}$. Therefore, (X, r_1C) is complete. The proof follows by transfinite induction. $\diamond$

Corollary 3.4 (X, rC) is complete, if (X, C) is complete.

Observe that $cl^n_{p_{r_1(k)}}f(\mathcal{F}) \leq f(cl^n_{p_{r_1C}}\mathcal{F})$, which establishes the continuity of f on (X, r_1C). The proof of the following proposition follows by applying induction and Proposition 3.2.

Proposition 3.5 If $f : (X, C) \to (Y, K)$ is continuous, then $f : (X, rC) \to (Y, rK)$ is also continuous.

The notion of s-map, introduced in [8] can be extended to filter spaces in general. A continuous map $f : (X, C) \to (Y, K)$ is an *s-map*, if $\mathcal{F} \in C$ converges to at most one point implies that $f(\mathcal{F}) \in K$ converges to at most one point. The identity map, any continuous map with a T_2 co-domain and the embedding map in a stable completion of a filter space and is an s-map. are all examples of s-maps. The $c-$ regylarity and $c-$ separatedness of Cauchy spaces [4] are generalised for filter spaces by using s-map.

Definition 3.6 A filter space (X, C) is said to be *c-regular* if $\mathcal{F} \notin C$ implies that $\exists$ a complete regular filter space (Y, K) and an s-map $f : (X, C) \to (Y, K)$ such that $f(\mathcal{F}) \notin K$.

Examples of c-regular filter spaces are not difficult to find out. In fact, each complete regular filter space is c-regular. This can be verified by taking $(Y, K) = (X, C)$ and $f = I_X$ in Definition 3.6.

Proposition 3.7 Every c-regular filter space is a c-filter space.

Proof. Let (X, C) be a c-regular filter space. Let $\mathcal{F} \in C$ and $\mathcal{F} \sim_c \dot{x}$. If possible, let $\mathcal{F} \cap \dot{x} \notin C$. So $\exists$ a complete regular filter space (Y, K) and an s-map $f : (X, C) \to (Y, K)$ such that $f(\mathcal{F} \cap \dot{x}) \notin K$. Since f is an s-map, $f(\mathcal{F}) \in K$ and $f(\mathcal{F}) \sim_k f(\dot{x})$. But, since every regular filter space is a c-filter space, (Y, K) is a c-filter space and this leads to a contradiction. $\diamond$

The proof of the following lemma has been omitted, since it can be proved the same way as the Lemma 3.15[8].

Lemma 3.8 A filter space (X, C) has a regular stable completion iff $((\tilde{X}, r\tilde{C}), j)$ is a regular completion of (X, C).

Proposition 3.9 A filter space (X, C) has a regular stable completion iff it is c-regular.

Proof. ($\Leftarrow$) Let (X, C) be c-regular. By Lemma 3.8, we need only show that $j^{-1} : (\tilde{X}, r\tilde{C}) \to (X, C)$ is continuous. If possible, let $\exists \mathcal{H} \in r\tilde{C}$ such that$j^{-1}(\mathcal{H}) \notin C$. By the c-regularity of (X, C), $\exists$ a complete regular space (Y, K) and an s-map $f : (X, C) \to (Y, K)$ such that $f(j^{-1}(\mathcal{H})) \notin K$. Since (Y, K), being regular, is a c-filter space it follows that the extension $\tilde{f} : (\tilde{X}, \tilde{C}) \to (Y, K)$ exists. Since (Y, K) is regular, $\tilde{f} : (\tilde{X}, r\tilde{C}) \to (Y, K)$ is continuous. Since $\mathcal{H} \in r\tilde{C}$, $\tilde{f}(\mathcal{H}) \in K$. But $f(j^{-1}(\mathcal{H})) \geq \tilde{f}(\mathcal{H})$ implies that $f(j^{-1}(\mathcal{H})) \in K$, a contradiction. So j^{-1} is continuous and therefore, $((\tilde{X}, r\tilde{C}), j)$ is a regular, stable completion.
($\Rightarrow$) By Lemma 3.8, $((\tilde{X}, r\tilde{C}), j)$ is a regular stable completion of (X, C). Since j^{-1} is continuous, $\mathcal{F} \notin C$ implies that $j(\mathcal{F}) \notin r\tilde{C}$. Also, since this is a stable completion, j is an s-map. Therefore, (X, C) is c-regular. $\diamond$

Let (X, C) and (Y, K) be c-regular filter spaces. By Proposition 3.7, (Y, K) is a c-filter space. Hence, if $f:(X,C)\to(Y,K)$ is an s-map, then by Proposition 2.2 in [3], there exists a unique s-extension $f^*:(X^*,rC^*)\to(Y^*,rK^*)$ such that $j_Y\circ f=f^*\circ j_X$.

Let $CRFIL$ be the full subcategory of FIL consisting of c-regular filter spaces as objects and s-maps as morphisms. Note that $CRFIL$ is a full subcategory of $CFIL'$ whose objects are all c-filter spaces and morphisms are the s-maps. Define for any object (X,C) in $CRFIL$, $R(X,C)=(X^*,rC^*)$ and for any morphism f, $R(f)=f^*$. Recall that $RFIL$ is the full subcategory of FIL consisting of all regular objects in FIL. If $CRFIL^*$ is a full subcategory of $CRFIL$ consisting of complete objects in $CRFIL$ and $RFIL^*$ is the full subcategory of $RFIL$ consisting of complete objects in $RFIL$, then $CRFIL^*=RFIL^*$.

Proposition 3.10 $R:CRFIL\to RFIL^*$ defined by $R(X,C)=(X^*,rC^*)$ for each object (X,C) in $CRFIL$, and $R(f)=f^*$ for each morphism f, is a completion functor and $CRFIL$ is a completion subcategory [8] of $CFIL'$.

The proof of this proposition is simple, since the functor $R:CRFIL\to RFIL^*$ is a reflector.

Definition 3.11 A filter space is *c-separated*, if it is T_2 and the following condition holds

for $\mathcal{F},\ \mathcal{G}\in C$ there is no $\mathcal{H}\in C$ such that $\mathcal{F}\vee\mathcal{H},\ \mathcal{H}\vee\mathcal{G}$ exist implies that $\exists$ a complete T_3 filter space (Y,K) and a continuous map $f:(X,C)\to(Y,K)$ such that $f(\mathcal{F})\not\sim_k f(\mathcal{G})$.

The proof of the following proposition is routine. **Lemma 3.12** The property c-separated is a hereditary property.

A C_3 filter space is one which admits a T_3 completion.

Proposition 3.13 Any C_3 filter space is c-regular and c-separated.

Proof. Let $((Y,K),\phi)$ be a T_3 completion of a filter space (X,C). Since $((Y,K),\phi)$ is a stable completion, by Proposition 3.9, (X,C) is c-regular. So it remains to show that (X,C) is c-separated. From Lemma 3.12 it follows that $(\phi(X),K_{(\phi(X))})$ is c-separated. Let $\mathcal{F},\ \mathcal{G}\in C$ such that there exists no $\mathcal{H}\in C$ for which $\mathcal{F}\vee\mathcal{H},\ \mathcal{H}\vee\mathcal{G}$ exist. Then there is no $\mathcal{T}\in K$ such that $\phi(\mathcal{F})\not\sim_K\phi(\mathcal{G})$ exist, and consequently, there is no $\mathcal{S}\in K_{\phi(X)}$ such that $\phi(\mathcal{F})\vee\mathcal{S},\ \mathcal{S}\vee\phi(\mathcal{G})$ exist. Hence, $\exists$ a complete T_3 filter space (Z,S) and a continuous map $g:(\phi(X),D')\to(Z,S)$ such that $g(\phi(\mathcal{F}))\not\sim_s g(\phi(\mathcal{G}))$. Since $f=g\circ\phi$ is a continuous map, (X,C) is c-separated. $\diamond$

Recall that $((X^*,C^*).j)$ [3] is the Wyler completion of a T_2 Cauchy space (X,C), where X^* is the set of all equivalence classes of filters in C, the mapping j is defined by $j(x)=[\dot{x}]$, $\forall x\in X$ and $C^*=\{\mathcal{A}\in\mathbf{F}(X^*)\mid$ either $\exists$ a convergent filter $\mathcal{F}\in C$ such that $\mathcal{A}\geq j(\mathcal{F})$ or $\exists$ a non convergent filter $\mathcal{G}\in C$ such that $\mathcal{A}\geq j(\mathcal{G})\cap[\dot{\mathcal{G}}]\}$.

Lemma 3.14. If (X,C) is c-separeted, then (X^*,rC^*) is T_2.

Proof. We argue contrapositively. Let $[\mathcal{F}],\ [\mathcal{G}]\in X^*$ such that $[\mathcal{F}]\neq[\mathcal{G}]$. We show that $[\dot{\mathcal{F}}]\sim_{rc^*}[\dot{\mathcal{G}}]$. Now $\mathcal{F}\not\sim_c\mathcal{G}$ implies that $\exists$ no $\mathcal{H}\in C$ such that $\mathcal{F}\vee\mathcal{H},\ \mathcal{H}\vee\mathcal{G}$ exist. So, $\exists$ a T_3 Cauchy space (Y,K) and a continuous map $f:(X,C)\to(Y,K)$ such that $f(\mathcal{F})\not\sim f(\mathcal{G})$. By Proposition 2.2 [KR], let $f^*:(X^*,C^*)\to(Y,K)$ be the unique extension of f such that $f^*\circ j=f$ and by Proposition 3.5, $f^*:(X^*,rC^*)\to(Y,K)$ is continuous. We have $f^*([\mathcal{F}])\neq f^*([\mathcal{G}])$, because $f(\mathcal{F})\not\sim_k f(\mathcal{G})$, and $f(\mathcal{F})=f^*\circ j(\mathcal{F})\overset{p_k}{\to}f^*([\mathcal{F}])$ and similarly $f(\mathcal{G})\overset{p_k}{\to}f^*([\mathcal{G}])$. Since (Y,K) is T_2, this implies that $f^*([\dot{\mathcal{F}}])\not\sim_k f^*([\dot{\mathcal{G}}])$. Since f^* is continuous, $[\dot{\mathcal{F}}]\not\sim_{rc^*}[\dot{\mathcal{G}}]$. This proves that (X^*,rC^*) is T_2. $\diamond$

The following proposition gives a characterization for C_3 filter spaces. The corresponding characterization for C_3 Cauchy spaces was studied in details by Kent and Richardson [4].

Proposition 3.15. A filter space is C_3 iff it is c-regular and c-separated.

Proof. $(\Rightarrow)$ Proof of this direction follows from Proposition 3.13.

$(\Leftarrow)$ Let (X,C) be a c-regular and c-separated filter space and $((X^*,C^*),j)$ be its Wyler completion. We show that $((X^*,rC^*),j)$ is a T_3 completion of (X,C). The fact that (X^*,rC^*) is complete and T_2, follows from Proposition 3.4 and 3.14 respectively. Therefore, it remains to show that $((X^*,rC^*),j)$ is a completion of (X,C). Since $((X^*,C^*),j)$ is a completion of (X,C) and $rC^*\leq C^*$ it follows that $((X^*,C^*),j)$ is complete and $j:(X,C)\to(X^*,rC^*)$ is continuous such that $X^*=cl_{p_{c^*}}j(X)\subseteq cl_{p_{rc^*}}j(X)$. Then, as in the later part of Proposition 3.9 we can show that j^{-1} is continuous. This proves Proposition 3.15. $\diamond$.

Corollary 3.16. If a T_2 filter space (X,C) admits a regular completion, then $((X^*,rC^*),j)$ is a regular completion of (X,C). $((X^*,rC^*),j)$ is a

T_3 completion iff (X,C) is c-separated.
The proof of the following proposition is now immediate.
Proposition 3.17 The following statements for a T_3 filter space (X,C) are equivalent:
(I) (X,C) is C_3,
(II) (X,C) has a regular completion and (X,C) is c-separated,
(III) $((X^*,rC^*),j)$ is a T_3 completion of (X,C).

The T_3 completions also have the extension properties similar to the regular completions. Let $CRSFIL$ be the full subcategory of T_2FIL consisting of all c-regular and c-separated filter spaces as objects and continuous maps as morphisms. It follows that $CRSFIL^* = T_3FIL^*$, where $CRSFIL^*$ consists of the complete objects in $CRSFIL$ and T_3FIL^* consists of the complete T_3 objects in T_2FIL.
Proposition 3.18 $R' : CRSFIL \to CRSFIL^*$ defined by $R'(X,C) = (X^*,rC^*)$, $\forall$ objects (X,C) in $CRSFIL$ and $R'(f) = f^*$, $\forall$ morphisms f, where f^* is the unique extension of f, is a completion functor. $CRSFIL$ is a completion subcategory of T_2FIL.
The proof of this proposition follows from the fact that the functor R' is a reflector. The following lemma is required to characterize the locally compact C_3 spaces.
Lemma 3.19 Let (X,C) be a locally compact T_3 filter space which is also c-separated. If $A \subseteq X^*$, then $j^{-1}(cl_{p_{rc^*}}A) = cl_{p_c}j^{-1}(A)$.
Proof. Let $x \in cl_{p_c}j^{-1}(A)$. So $\exists\ \mathcal{F} \in C$ such that $\mathcal{F} \overset{p_c}{\to} x$ and $j^{-1}(A) \in \mathcal{F}$. This implies that $j(\mathcal{F}) \overset{p_{rc^*}}{\to} j(x)$ and $A \in j(\mathcal{F})$. Therefore, $x \in j^{-1}cl_{p_{rc^*}}A$ and so $cl_{p_c}j^{-1}(A) \subseteq j^{-1}(cl_{p_{rc^*}}A)$. To prove the inclusion in the other direction, let $x \in j^{-1}(cl_{p_{rc^*}}A)$. So $\exists\ \mathcal{A} \in rC^*$ such that $\mathcal{A} \overset{p_{rc^*}}{\to} j(x)$ and $A \in \mathcal{A}$. $\exists\ \mathcal{F} \in C$ and some $n \in N$ such that $\mathcal{A} \geq cl^n_{p_{rc^*}}j(\mathcal{F}) \cap [\dot{\mathcal{F}}]$. This implies that $\mathcal{A} \overset{p_{rc^*}}{\to} [\mathcal{F}]$. By Proposition 3.14, $\mathcal{F} \sim_c \dot{x}$ and since (X,C) is regular, $\mathcal{F} \cap \dot{x} \in C$, which implies that $\mathcal{A} \geq cl^n_{p_{rc^*}}j(\mathcal{F}\cap\dot{x})$. Since $cl_{p_c}(\mathcal{F}\cap\dot{x})$ has a base of compact sets and since (X^*,rC^*) is T_2, these compact sets are closed. So $cl^n_{p_{rc^*}}j(\mathcal{F}\cap\dot{x}) = j(cl_{p_c}\mathcal{F}\cap\dot{x})$, which implies that $\mathcal{A} \geq j(cl_{p_c}\mathcal{F}\cap\dot{x})$. Moreover since $\mathcal{F} \overset{p_c}{\to} x$, $cl_{p_c}(\mathcal{F}\cap\dot{x})$ also converges to x. Hence $cl_{p_c}(\mathcal{F}\cap\dot{x})$ has a trace $\mathcal{G}$ on $j^{-1}(A)$ and $\mathcal{G} \overset{p_c}{\to} x$. This implies that $x \in cl_{p_c}j^{-1}(A)$, which proves the lemma. $\diamond$
Proposition 3.20 A locally compact T_3 filter space is C_3 iff (X,C) is c-separated.
Proof. ($\Leftarrow$) This follows from Proposition 3.13.
($\Rightarrow$) Let (X,C) be c-separated. We show that $((X^*,rC^*),j)$ is a T_3 completion of (X,C). By Lemma 3.14, (X^*,rC^*) is T_2 and by Lemma 3.19, $j^{-1}(cl^n_{rc^*}j(\mathcal{F})) = cl^n_{p_c}\mathcal{F}$ for all $n \in N$ and $\mathcal{F} \in C$. Let $rC^*_{j(X)}$ be the subspace structure on $j(X)$. So it follows that $j^{-1} : (j(X),rC^*_{j(X)}) \to (X,C)$ is a continuous map. Therefore, (X^*,rC^*) is a T_3 completion of (X,C). This proves Proposition 3.20. $\diamond$

4 Strict completions

In this section, we assume that (X,C) is a T_2 filter space, unless otherwise mentioned. Kent and Richardson [4] obtained strict T_3 completions for Cauchy spaces. Similar techniques are used to obtain strict completions for filter spaces.
Definition 4.1 A completion $((Y,K),\psi)$ of a filter space (X,C) is said to be *strict* if $\mathcal{A} \in K$ implies that $\exists\ \mathcal{F} \in C$ such that $\mathcal{A} \geq cl_{p_k}\psi(\mathcal{F})$.

It is a known that all topological completions are strict. Also if (X,C) is a T_2 filter space and $X^* \setminus j(X)$ is finite, where $((X^*,C^*),j)$ is the T_2-Wyler completion of (X,C), then any completion of (X,C) is strict. Let $\Sigma A = \{[\mathcal{F}] \in X^* \mid A \in \mathcal{G}$ forsome $\mathcal{G} \in [\mathcal{F}]\}$ for each subset $A \subseteq X$ and $\Sigma\mathcal{F} = [\Sigma F \mid F \in \mathcal{F}\}]$, for each $\mathcal{F} \in \mathbf{F}(X)$. Let $C^*_1 = \{\mathcal{A} \in \mathbf{F}(X^*) \mid \exists \mathcal{F} \in C$ such that $\mathcal{A} \geq \Sigma\mathcal{F}\}$.
Proposition 4.2. $((X^*,C^*_1),j)$ is a completion of (X,C) in standard form [KR] iff (X,C) is regular.
Proof. Let (X,C) be regular. C^*_1 is a filter structure on X^*, because $[\dot{\mathcal{F}}] \geq \Sigma\mathcal{F}$. The mapping j as defined in the T_2-Wyler completion is injective, and since $j(\mathcal{F}) \geq \Sigma\mathcal{F}$, $\forall\mathcal{F} \in C$, j is a continuous map. (X^*,C^*_1) is complete, since $\Sigma\mathcal{F} \overset{p_{c^*_1}}{\to} [\mathcal{F}]$. Observe that for $A \subseteq X$, $cl_{p_c}A = j^{-1}\Sigma A$. So it follows that for any $\mathcal{A} \in C^*_1$, with $\mathcal{A} \geq \Sigma\mathcal{F}$ $j^{-1}\mathcal{A} \geq cl_{p_c}\mathcal{F} \in C$. Therefore, j^{-1} is a continuous map. Also, since $C^*_1 \leq C^*$, (X^*,C^*_1) is a completion of (X,C). Also it is a completion in standard form, since $j(\mathcal{F}) \geq \Sigma\mathcal{F} \overset{p_{c^*_1}}{\to} [\mathcal{F}]$.

Conversely, if $((X^*,C^*_1),j)$ is a completion of (X,C), then $\Sigma\mathcal{F} \in C^*_1$, $\forall\mathcal{F} \in C$, which implies that $j^{-1}\Sigma\mathcal{F} = cl_{p_c}\mathcal{F} \in C$. So (X,C) is regular.
This proves Proposition 4.2. $\diamond$
The following proposition shows that $((X^*,C^*_1),j)$ is the unique strict completion for a T_2 filter space

(X, C).

Proposition 4.3.

(I) If $((X^*, K), j)$ is a T_2 completion of (X, C) in standard form, then for any subset $A \subseteq X$ $cl_{p_k} j(A) = \Sigma A$.

(II) A completion $((X^*, K), j)$ is strict iff $K \geq C_1^*$.

Proof. (I) $[\mathcal{F}] \in \Sigma A \Rightarrow \exists \mathcal{G} \in C$ such that $\mathcal{G} \in [\mathcal{F}]$ and $A \in \mathcal{G}$. So $j(\mathcal{G}) \sim_k j(\mathcal{F})$ and since $((X^*, K), j)$ is in standard form, $j(\mathcal{G}) \overset{p_k}{\to} [\dot{\mathcal{F}}]$. Also, $j(A) \in j(\mathcal{G})$, which shows that $[\mathcal{F}] \in cl_{p_k} j(A)$. Next, let $[\mathcal{F}] \in cl_{p_k} j(A)$. $\exists\ \mathcal{H} \in K$ such that $\mathcal{H} \overset{p_k}{\to} [\dot{\mathcal{F}}]$ and $j(A) \in \mathcal{H}$. Since $\mathcal{G} = j^{-1}(\mathcal{H}) \in C$, $A \in \mathcal{G}$, and $j(\mathcal{G}) \sim_k [\dot{\mathcal{F}}]$, it follows that $\mathcal{G} \sim_c \mathcal{F}$. Hence $[\mathcal{F}] = [\mathcal{G}] \in \Sigma A$. Therefore, $\Sigma A = cl_{p_k} j(A)$.

(II) Let $((X^*, K), j)$ be a strict completion of (X, C). $\mathcal{A} \in K \Rightarrow \exists \mathcal{F} \in C$ such that $cl_{p_k} j(\mathcal{F}) \leq \mathcal{A}$. By part (I) it follows that $\mathcal{A} \in C_1^*$. Conversely, let $K \geq C_1^*$. $\mathcal{A} \in C_1^* \Rightarrow \exists \mathcal{F} \in C$ such that $\mathcal{A} \geq cl_{p_k} j(\mathcal{F})$. Hence $((X^*, C_1^*), j)$ is a strict completion of (X, C). This proves Proposition 4.3. $\diamond$

The proof of the following proposition now becomes routine.

Proposition 4.4. If (X, C) is a T_3 filter space, then $((X^*, C_1^*), j)$ is its unique strict completion in standard form.

Next, we define the Σ^2 operator on subsets $A \subseteq X$,

$\Sigma^2 A = \{[\mathcal{F}] \in X^* \mid \exists\ \mathcal{G} \in [\mathcal{F}]$ such that $\Sigma G \cap \Sigma A \neq \Phi\ \forall G \in \mathcal{G}\}$ and for $\mathcal{F} \in \mathbf{F}(X)$, $\Sigma^2 \mathcal{F} = [\{\Sigma^2 F \mid F \in \mathcal{F}\}]$. Note that for any $A \subseteq X$, $\Sigma^2 A \subseteq cl_{p_{c_1^*}} \Sigma A$. The following proposition gives a characterization for a T_3 filter space to have a strict T_3 completion.

Proposition 4.5. A T_3 filter space has a strict T_3 completion iff the following two conditions hold :

(I) If for any two filters $\mathcal{F}$ and $\mathcal{G}$ in C $\exists\ \mathcal{H}_1, \cdots, \mathcal{H}_n \in C$ such that $\Sigma\mathcal{F} \vee \Sigma H_1, \cdots, \Sigma\mathcal{H}_n \vee \Sigma\mathcal{G}$ exist, then $\mathcal{F} \sim_c \mathcal{G}$.

(II) $\mathcal{F} \in C$ implies that $\Sigma^2 \mathcal{F} \in C_1^*$.

Proof. $(\Rightarrow)$ Since (X, C) is T_3, by Proposition 4.3, $((X^*, C_1^*), j)$ is its unique strict T_3 completion. To prove condition (I), let for $\mathcal{F}$, $\mathcal{G} \in C$ $\exists\ \mathcal{H}_1, \cdots, \mathcal{H}_n \in C$ such that $\mathcal{F} \vee \mathcal{H}_1, \cdots, \mathcal{H}_n \vee \mathcal{G}$ exist. Since $\Sigma H_i \in C_1^*$, $\forall\ i = 1, \cdots, n$, and $[\dot{\mathcal{F}}] \geq \Sigma\mathcal{F}$, it follows that $[\dot{\mathcal{F}}] \sim_{c_1^*} [\dot{\mathcal{G}}]$. Since C_1^* is T_2, $\mathcal{F} \sim_c \mathcal{G}$. Now for any filter $\mathcal{F} \in C$, $\Sigma\mathcal{F} \in C_1^*$, and since (X^*, C_1^*) is regular, $cl_{p_{c_1^*}} \Sigma\mathcal{F} \in C_1^*$. So $\Sigma^2 \mathcal{F} \geq cl_{p_{c_1^*}} \Sigma\mathcal{F}$ is in C_1^*.

$(\Leftarrow)$ By Proposition 4.3, it remains to show that (X^*, C_1^*) is a T_3 space. To show that this space is T_2, let $[\dot{\mathcal{F}}] \sim_{c_1^*} [\dot{\mathcal{G}}]$. So $\exists\ \mathcal{H}_1, \cdots \mathcal{H}_n \in C$ such that $\Sigma\mathcal{F} \vee \Sigma\mathcal{H}_1, \cdots, \mathcal{H}_n \vee \Sigma\mathcal{G}$ exist. By condition (I) this implies that $\mathcal{F} \sim_c \mathcal{G}$. Hence (X^*, C_1^*) is T_2. By condition (II), since $\Sigma^2\mathcal{F} \in C_1^*$, $cl_{p_{c_1^*}} \Sigma\mathcal{F} \in C_1^*$ for each $\Sigma\mathcal{F} \in C_1^*$. Therefore, (X^*, C_1^*) is regular. This completes the proof of Proposition 4.5. $\diamond$

The completion $((X^*, C_1^*), j)$ also has the universal property same as the Wyler completion. The completely regular and w-regular filter spaces are still to be studied in a subsequent paper.

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