A Complex Lyapunov Theory-based Adaptive Algorithm For Complex Signal Processing

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Abstract: - This paper presents a complex-valued version of the Lyapunov adaptive filtering algorithm [1]. The resulting algorithm simultaneously updates the real and imaginary parts of the complex coefficients so that the complex error can converge to zero asymptotically. The proposed scheme can be applied to random and deterministic processes because only the desired signal and input signal are required. The design is independent of the stochastic properties of signals and the stability is guaranteed by the Lyapunov Stability Theory (LST). This scheme possesses distinct advantages of stability, speed of convergence, computational complexity and robustness to additive noise or disturbance over some complex adaptive algorithms. Simulation examples are included to demonstrate the performance of the new complex adaptive algorithm.

Key-Words: Adaptive filtering, complex signal processing, Lyapunov stability theory

1 Introduction

Most available adaptive filters are real-valued and are suitable for signal processing in real-dimensional space. In some applications, signals are complex-valued and processing is done in complex-dimensional space. An example is the channel equalization of communication channels with complex signaling schemes such as quadrature amplitude modulation (QAM). Another application example is the frequency domain adaptive filtering [12] where the signals and filter coefficients are complex. For complex signal processing problems, many existing adaptive algorithms cannot be applied directly. Although for certain applications it is possible to reformulate a complex signal processing problem so that a real-valued adaptive algorithm can be used to solve the problem, it is not always feasible to do so and it is preferred to preserve the concise formulation and elegant structure of complex signals. [3]

Some researchers proposed the complex adaptive algorithms by extending the real-version adaptive algorithms such as LMS (*Least Mean Square*) and RLS (*Recursive Least Square*) to the complex forms [3]-[5]. For complex-LMS, the mean squared value of the complex error is minimized [5]. Widrow [5] showed that the weighting is controlled by the complex-LMS adaptation *step-size*. A large *step-size* leads to rapid convergence but the filter parameters may oscillate or become unstable, while low value implies slow convergence. [4] Real-RLS is computationally expensive to implement even with the availability of the fast algorithm and it exhibits unstable performance [6]. Methods of avoiding instability have been proposed in [7]-[10] but the stability problem of the adaptive filters have not been solved if there are some bounded input disturbances. Similar problems are encountered by the complex-RLS.

Authors [1] have noticed the stability problem of real-RLS adaptive filter and proposed Lyapunov theory-based adaptive filtering (LAF). The design adaptive filter is the modification of RLS algorithm using LST. The LAF [1],[2] is independent of the stochastic properties of the signals. Based on the observations and a collection of desired response, the filter coefficients are updated in the Lyapunov sense so that the error between the desired response and the filter output can asymptotically converge to zero. It has been shown in [1] that, unlike many adaptive filtering schemes using gradient search in the parameter space, LAF algorithm uses a Lyapunov function $V(k)$, which is positive definite, with a unique global minimum in the state space. By properly choosing the parameter update law in the sense that $\Delta V(k) = V(k) - V(k-1)$ is negative, the output of the adaptive filter can asymptotically converge to the desired reference signal according to LST [11]. Therefore, the local minima problem occurred in the gradient search based adaptive filters is avoided and at the same time, the stability of the error dynamics is guaranteed. The LAF in [1] is only designed for real-value adaptive filtering and it cannot be applied to complex-value applications directly. Therefore, a complex version of the LAF algorithm is proposed in this paper.

2 Problem Formulation

The typical structure of an adaptive filtering system is illustrated in Figure 1. For real signal processing, $x(k)$

is the input real signal of the filter, which has been disturbed by the nonlinearity of the communication channel and noises, $y(k)$ is the real output signal of the filter, $d(k)$, the real *desired response*, is provided for the output of the filter to follow, $e(k)$ is the real error between the desired reference signal $d(k)$ and the output of the filter $y(k)$.

$$e(k) = d(k) - y(k) \qquad (2.1)$$

The adaptive algorithm in Figure 1 is generally designed to update the filter real coefficients so that the cost function of the error is minimized in the parameter space.

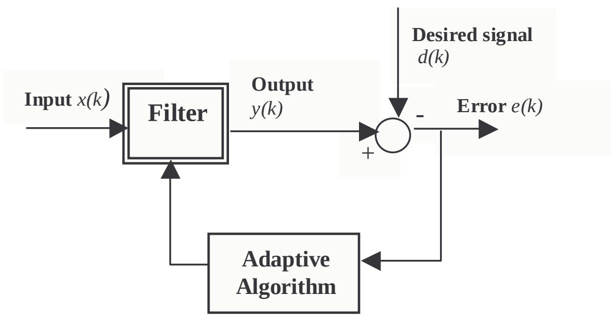

Figure 1: Adaptive Filtering Problem

For complex signal processing,

$$\begin{aligned} \boldsymbol{e}(k) &= \boldsymbol{d}(k) - \boldsymbol{y}(k) \qquad (2.2)\\ &= Re[\boldsymbol{e}(k)] + j\, Im[\boldsymbol{e}(k)] \\ &= |\boldsymbol{e}(k)|\, arg(\boldsymbol{e}(k)) \end{aligned}$$

where

$$\begin{aligned} y(k) &= Re[\boldsymbol{y}(k)] + j\, Im[\boldsymbol{y}(k)] \\ &= \boldsymbol{H}^T(k)\boldsymbol{X}(k) \qquad (2.3) \end{aligned}$$

The complex input vector $\boldsymbol{X}(k)$ and complex coefficient vector $\boldsymbol{H}(k)$ are given by

$$\boldsymbol{H}(k) = Re[\boldsymbol{H}(k)] + j\, Im[\boldsymbol{H}(k)] \qquad (2.4)$$

and

$$\boldsymbol{X}(k) = Re[\boldsymbol{X}(k)] + j\, Im[\boldsymbol{X}(k)] \qquad (2.5)$$

3 Design of Complex LAF Adaptive Algorithm Using LST

The design of the complex adaptive filter is described by *Theorem 3.1*:

Theorem 3.1: For the given input signal, if the filter complex coefficients $\boldsymbol{H}(k)$ of the filter, $y(k) = \boldsymbol{H}^T(k)\boldsymbol{X}(k)$ is updated as follow

$$\boldsymbol{\alpha}(k) = \boldsymbol{d}(k) - \boldsymbol{H}^T(k-1)\boldsymbol{X}(k) \qquad (3.1)$$

$$\boldsymbol{g}(k) = \frac{\boldsymbol{X}^*(k)}{\boldsymbol{X}^T(k)\boldsymbol{X}^*(k)}\left(1-\kappa\frac{|\boldsymbol{e}(k-1)|}{\boldsymbol{\alpha}(k)}\right) \qquad (3.2)$$

$$\boldsymbol{H}(k) = \boldsymbol{H}(k-1) + \boldsymbol{g}(k)\boldsymbol{\alpha}(k) \qquad (3.3)$$

where $\boldsymbol{g}(k)$ is the complex adaptation gain, $\boldsymbol{\alpha}(k)$ is the complex a priori estimation error, and $0 \le \kappa < 1$, then the complex error $\boldsymbol{e}(k)$ asymptotically converges to zero.

*Note: * : complex conjugate, ^{T}: transpose*

Proof: Define a Lyapunov function of the complex error $\boldsymbol{e}(k)$

$$\begin{aligned} V(k) &= \boldsymbol{e}(k)\boldsymbol{e}^*(k) \\ &= |\boldsymbol{e}(k)|^2 \qquad (3.4) \end{aligned}$$

$$\begin{aligned} \Delta V(k) &= V(k) - V(k-1) \\ &= |e(k)|^2 - |e(k-1)|^2 \\ &= |d(k) - H^T(k)X(k)|^2 - |e(k-1)|^2 \\ &= |d(k) - (H^T(k-1) + g^T(k)\alpha(k))X(k)|^2 \\ &\quad - |e(k-1)|^2 \\ &= |d(k) - H^T(k-1)X(k) - g^T(k)\alpha(k)X(k))|^2 \\ &\quad - |e(k-1)|^2 \\ &= |\alpha(k) - g(k)\alpha(k)X(k)|^2 - |e(k-1)|^2 \end{aligned} \qquad (3.5)$$

Using the expression (3.2) in the expression (3.5), we have

$$\begin{aligned} \Delta V(k) &= |\kappa e(k-1)|^2 - |e(k-1)|^2 \\ &= -(1-\kappa^2)|e(k-1)|^2 < 0 \qquad (3.6) \end{aligned}$$

Remark 3.1: According to Theorem 3.1, the stability of the complex error dynamics $\boldsymbol{e}(k)$ can be guaranteed based on the LST [11]. The convergence rate of complex error $\boldsymbol{e}(k)$ depends on κ. It is easy to see that convergence analysis of the complex LAF algorithm is similar to the real LAF algorithm given in [1] and can be carried out easily.

Remark 3.2: The adaptive gain, $g(k)$ can be modified as the expression in (3.7) respectively to prevent the singularities due to zero values of $\boldsymbol{\alpha}(k)$ and $\boldsymbol{X}^T(k)\boldsymbol{X}^*(k)$.

$$\boldsymbol{g}(k) = \frac{\boldsymbol{X}^*(k)}{\boldsymbol{X}^T(k)\boldsymbol{X}^*(k)+\lambda_1}\left(1-\kappa\frac{|\boldsymbol{e}(k-1)|}{\boldsymbol{\alpha}(k)+\lambda_2}\right) \qquad (3.7)$$

where λ_1, λ_2 are small complex numbers, for example, $\lambda_1 = \lambda_2 = 0.001+j0.001$. These constants can be chosen to have same values. Smaller these values contribute smaller complex error, $\boldsymbol{e}(k)$.

4 Simulation Example

The performance of the proposed complex-LAF adaptive algorithm is illustrated in the example of adaptive filtering for complex signal. In this simulation, the complex desired response of the adaptive filter is defined as

$$\boldsymbol{d}(k) = e^{(-0.01k(1-2j))} + (1+j)$$

and the filter complex input signal ***x(k)*** which is corrupted by the additive complex noise, is given by

$$\boldsymbol{x}(k) = \boldsymbol{d}(k) + \boldsymbol{w}(k)(1+j)$$

where ***w(k)***(1+j) is a bounded complex random noise that satisfies the following bounded condition

$$0 \le w(k) \le 0.1.$$

The adaptive filter has the following structure:

$$y(k) = h_k(0)x(k) + h_k(1)x(k-1) + h_k(2)x(k-2)$$

and the adaptive gain is updated according to the expression (3.7). Initially, the parameters λ_1, λ_2 and κ in expression (3.7) are chosen as follow:

$$\lambda_1 = \lambda_2 = 0.1 + 0.1j, \text{ and } \kappa = 0.1$$

Fig. 2a and **Fig. 2b** show the reference signal ***d(k)*** and the corrupted filter input signal ***x(k)*** in complex plane respectively. The filter complex output, ***y(k)*** is shown in **Fig. 2c**. **Fig. 2d** shows the complex error, ***e(k)***. It is seen that, although the output of the adaptive filter can follow the desired reference signal very well, the effects of the noise are not fully eliminated because the adaptation rate is relatively slow ($\kappa = 0.1$) and the values of the parameters λ_1, λ_2 are very large.

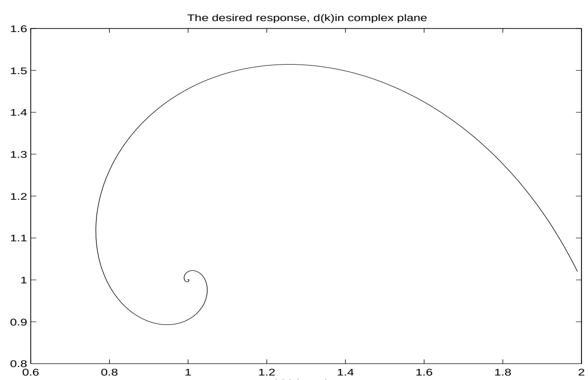

Fig. 2a: The desired response, ***d(k)*** in complex plane

In the second case, the parameters λ_1, λ_2 and κ in expression (3.7) are selected as follow

$$\lambda_1 = \lambda_2 = 0.01 + 0.01j, \text{ and } \kappa = 0.01$$

Fig. 3a and **Fig. 3b** illustrate the filter output, *y(k)* and the tracking error, ***e(k)***. **Fig. 3c** and **Fig. 3d** show the real and imaginary adaptive filter coefficients, *Re*[***H(k)***] and *Im*[***H(k)***] respectively.

It can be seen that the effect of the input disturbance has been greatly reduced and the tracking performance between ***d(k)*** and ***y(k)*** of the adaptive filter has been greatly improved by properly choosing parameters λ_1, λ_2 and κ.

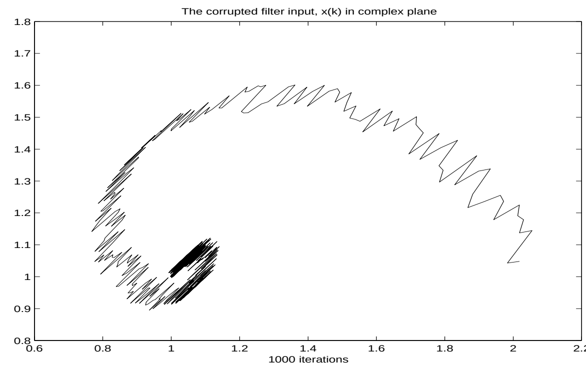

Fig. 2b: The corrupted filter input, ***x(k)*** in complex plane

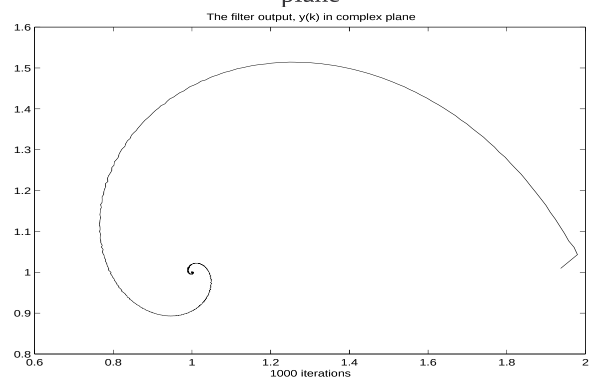

Fig. 2c: The filter output, ***y(k)*** in complex plane

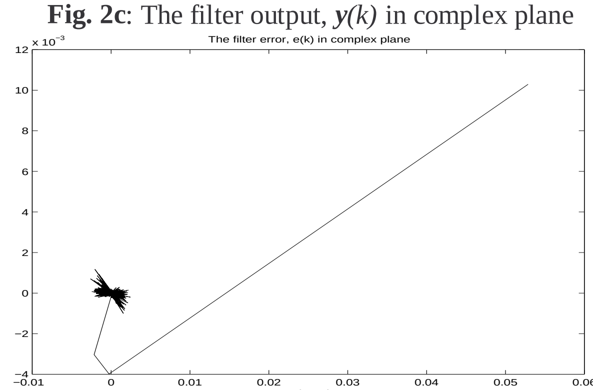

Fig. 2d: The filter error, ***e(k)*** in complex plane

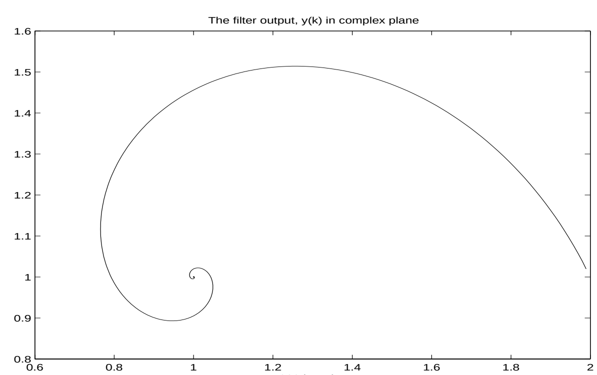

Fig. 3a: The filter output, ***y(k)*** in complex plane

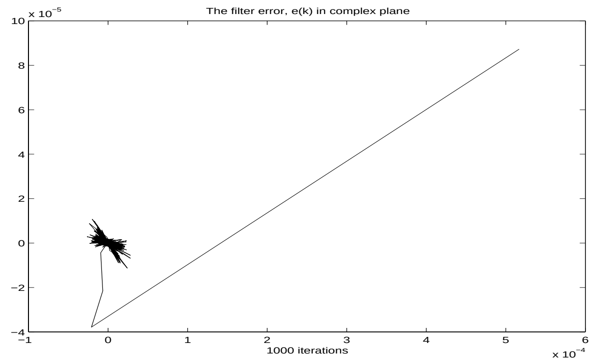

Fig. 3b: The filter error, ***e****(k)* in complex plane
(Note: y-axis: $x10^{-5}$, x-axis: $x10^{-4}$)

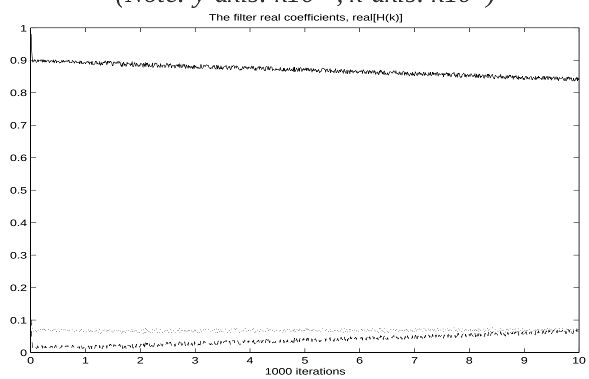

Fig. 3c: The filter real coefficients, Re[***H****(k)*]

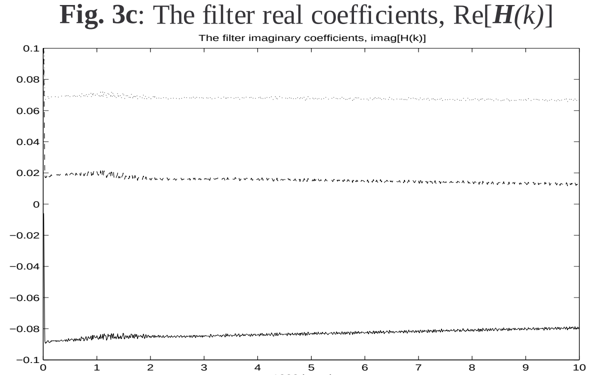

Fig. 3d: The filter imaginary coefficients, Im[***H****(k)*]

5 Conclusion

This paper has provided a new approach in designing a complex adaptive algorithm using the Lyapunov Stability Theory. The design of this complex filter is independent of stochastic properties of the signals The proposed scheme can be applied to random and deterministic processes because only the desired signal and input signal are required. The stability of this scheme is guaranteed by the Lyapunov stability theory. This scheme possesses distinct advantages of stability, speed of convergence, computational complexity and robustness to additive noise or disturbance over some complex adaptive algorithms. Simulation examples have revealed the performance that can be achieved based on the new complex adaptive algorithm.

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