On the independent set of de Bruijn graphs (Extended Abstract)

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Abstract: - The purpose of this study is to investigate the independent set of de Bruijn graphs. We propose an algorithm to obtain a cover of the de Bruijn graph $UB(d, D)$. For the investigation, we can use some properties concerning line digraph operation and matching, and cycle decomposition of de Bruijn digraphs.

Key-Words: - de Bruijn digraphs, independent set, line graph, cycle decomposition

1 Introduction

We construct an independent set of the de Bruijn graph using cycle decomposition and line digraph operation. In this paper, de Bruijn graphs are underlying graphs of de Bruijn digraphs.

$V(G)$ and $A(G)$ are the vertex set and the arc set of a digraph $G(V, A)$, respectively. The arcs are directed edges. There is an arc from x to y if $\langle x, y\rangle \in A(G)$. The vertex x is called a *predecessor of* y and y is called a *successor of* x. The pair of arcs $\langle x, y\rangle$ and $\langle y, x\rangle$ is called *symmetric arcs*. The vertices x and y are adjacent if $\langle x, y\rangle \in A(G)$ or $\langle y, x\rangle \in A(G)$. The sets $O(u) = \{v|\langle u, v\rangle \in A(G)\}$ and $I(u) = \{v|\langle v, u\rangle \in A(G)\}$ are called the *outset* and the *inset* of the vertex u, respectively.
For a digraph G with arcs, the *line digraph* $L(G)$ is the graph whose vertices correspond to the arcs of G, and for two vertices $\langle u_1, u_2\rangle$, $\langle v_1, v_2\rangle \in V(L(G))$, $\langle u_1, u_2\rangle$ is adjacent to $\langle v_1, v_2\rangle$ if and only if $u_2 = v_1$.
$V(G)$ and $E(G)$ are the vertex set and the edge set of a graph $G(V, A)$, respectively. The vertices x and y are adjacent if $\langle x, y\rangle \in E(G)$.

The *underlying graph* $U(G)$ of a digraph G is the graph obtained by replacing each arc $\langle u, v\rangle$ or symmetric arcs $\langle u, v\rangle$, $\langle v, u\rangle$ by an edge $\langle u, v\rangle$ and omitting loops.
The set $S \subset V(G)$ is *independent* if no two vertices in S are adjacent in the graph G. The independent set $S \subset V(G)$ is a *maximal independent set* if there is a vertex $v \in S$ such that u is adjacent to v, for any vertex $u \in V(G)\backslash S$. The maximal independent set is also called a *cover*. If a vertex set S is a cover, then S is also an independent dominating set[1]. The *independence number* $\beta_0(G)$ is the maximum cardinality of an independent set in a graph G. A maximum independent set is called a β_0-set.

The arcs $\langle u_1, u_2\rangle$ and $\langle v_1, v_2\rangle$ are adjacent if $u_i = v_j$ for some integers i and j $(1 \leq i, j \leq 2)$. A set $T \subset A(G)$ is *arc independent*, also called a *matching*, if no two arcs in T are adjacent. The *matching number* $\beta_1(G)$ is the maximum cardinality of an independent arc set in G. A maximum independent arc set is called a β_1-set.
The de Bruijn digraphs have several definitions. The definitions by alphabet, line digraph and modular arithmetic are well-known[2]. Now we state the definitions by alphabet and line digraph. Especially, the definition by line digraph is a key to construct independent sets of de Bruijn graphs in this paper. The definition of the de Bruijn digraph by alphabet is as follows: $V(B(d, D)) = Z_d^D$, Z_d^D is the set

of d-ary D dimensional vectors. $A(B(d,D)) = \{\langle(v_0,v_1,\ldots,v_{D-1}),(v_1,v_2,\ldots v_{D-1},x)\rangle | x \in Z_d\}$. We denote an edge by $\langle v_0,v_1,\ldots,v_{D-1},x\rangle$ instead of $\langle(v_0,v_1,\ldots,v_{D-1}),(v_1,v_2,\ldots v_{D-1},x)\rangle \in A(B(d,D))$. The definition of the de Bruijn digraph by line digraph is as follows: The de Bruijn digraph $B(d,1)$ is the complete symmetric digraph with loops K_d^+. $B(d,D)$ is $L(B(d,D-1))$. The de Bruijn digraph $B(2,3)$ is shown in Fig. 1. We can obtain the de Bruijn graph $UB(d,D)$ from de Bruijn digraph $B(d,D)$ by altering arcs to edges and symmetric arcs to one edge and omitting loops, i.e. $UB(d,D)$ is the underlying graph of de Bruijn digraph $B(d,D)$. Fig. 2 shows the de Bruijn digraph $B(2,4)$ and de Bruijn graph $UB(2,4)$

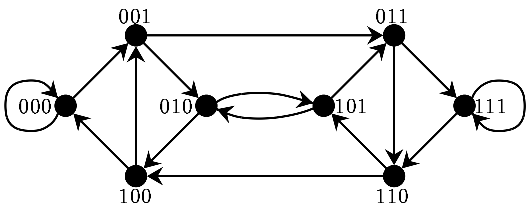

Fig. 1: de Bruijn digraph $B(2,3)$.

There are several studies for $UB(2,D)$ concerning independence and related topics. Livingston and Stout [6] studied the perfect dominating sets for binary de Bruijn digraphs $B(2,D)$ and binary de Bruijn graphs $UB(2,D)$. They showed that $UB(2,D)$ has a perfect dominating set for $D = 1,2$ but has no perfect dominating set for $D = 3,4,5$. Bryant and Fredricksen[3] demonstrated the bounds for the size of a cover of $UB(2,D)$. Lu et al.[7] investigated $(d,2)$-dominating number of $UB(2,D)$. Above studies deal with only binary de Bruijn graphs. The de Bruijn digraph $B(d,D)$ is d-regular, that is, $|O(u)| = |I(u)| = d$ for any $u \in V(B(d,D))$, but the de Bruijn graph is not a regular graph (see Fig. 2). This fact makes the study for de Bruijn graph difficult. We deal with the d-ary de Bruijn graph $UB(d,D)$. Since $B(d,D)$ is d-regular,the vertices without the vertices that have either self-loop or symmetric arcs have $2d$ neighbors in $UB(d,D)$ where $D \geq 3$.
The vertices that have self-loop in $B(d,D)$ have $2d-2$ neighbors in $UB(d,D)$ where $D \geq 3$. Thus we obtain the following proposition for lower and

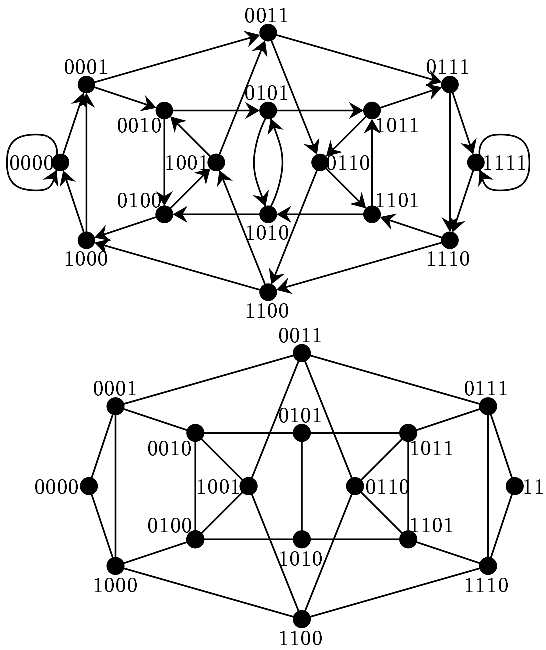

Fig. 2: de Bruijn digraph $B(2,4) = L(B(2,3))$ and de Bruijn graph $UB(2,4)$.

upper bounds for β_0.

Proposition 1.1 *For $UB(d,D)$ where $D \geq 3$, it follows that*

$$\frac{d^D}{2d+1} \leq \beta_0 \leq \frac{d^D}{2d-1}.$$

The *directed cycle* dC_n has n vertices, $|O(u)| = |I(u)| = 1$ for all vertex u and is a simple connected digraph with $|V(C_n)| = |E(C_n)|$ that can be drawn so that all of its vertices and arcs lie on a circle. It is easy to see that $L(dC_n)$ is isomorphic to dC_n. The *cycle* C_n is the underlying graph of dC_n.
A sequence $H_1, H_2, \ldots, H_n$ of graphs(digraphs) whose union is the graph(digraph) G is called a *decomposition* of G if each edge of G is in H_i for exactly one i $(1 \leq i \leq n)$, and in this case we write $G = H_1 \oplus H_2 \oplus \cdots \oplus H_n = \bigoplus_{i=1}^{n} H_i$. If each H_i $(1 \leq i \leq n)$ is a cycle(directed cycle), we call it a *cycle decomposition* of G. If in addition the subgraphs(subdigraphs) H_i are isomorphic to H, then we write $G = nH$, say that H *isomorphically decomposes* G, and write $H||G$.

We call a subgraph(subdigraph) F of G a *factor* of G if it contains all vertices of G. A sequence $F_1, F_2, \ldots, F_n$ of factors with $G = F_1 \oplus F_2 \oplus \cdots \oplus F_n = \bigoplus_{i=1}^{n} F_i$ is called a *factorization* of G.

Let a and b be integers. Then the greatest common divisor of a and b is written by $\gcd(a, b)$.

2 Cycle decomposition of de Bruijn digraphs

We introduce a function $\boldsymbol{f}_\sigma$ from Z_d^D into Z_d. For $\boldsymbol{f} = (f_1,\ f_2,\ f_3, \ldots,\ f_D,\ f') \in Z_d^{D+1}$, $\boldsymbol{a} = (a_1,\ a_2,\ a_3,\ \ldots,\ a_D) \in Z_d^D$ and any integer σ_i between 0 and $d-1$, $(0 \le i \le d-1)$, $\boldsymbol{f}_\sigma(\boldsymbol{a})$ is defined as follows:

$$\boldsymbol{f}_\sigma(\boldsymbol{a}) = \begin{cases} \sigma_0 & \text{if } f_1a_1 + f_2a_2 + \cdots + f_Da_D + f' \equiv 0 \pmod d \\ \sigma_1 & \text{if } f_1a_1 + f_2a_2 + \cdots + f_Da_D + f' \equiv 1 \pmod d \\ \sigma_2 & \text{if } f_1a_1 + f_2a_2 + \cdots + f_Da_D + f' \equiv 2 \pmod d \\ \vdots & \vdots \\ \sigma_d - 1 & \text{if } f_1a_1 + f_2a_2 + \cdots + f_Da_D + f' \equiv d-1 \pmod d \end{cases}$$

where σ is a mapping on Z_d and we could write σ out by showing what it does to every element, e.g., $\sigma : 0 \longrightarrow \sigma_0,\ 1 \longrightarrow \sigma_1,\ \ldots,\ d-1 \longrightarrow \sigma_{d-1}$. But this notation is cumbersome. Our short cut might be to write σ out as

$$\begin{bmatrix} 0 & 1 & 2 & \ldots & d-1 \\ \sigma_0 & \sigma_1 & \sigma_2 & \ldots & \sigma_{d-1} \end{bmatrix},$$

or more simply $[\sigma_0\ \sigma_1\ \sigma_2 \ldots \sigma_{d-1}]$ where σ_i is the image of i under σ. If $\sigma_i \ne \sigma_j$, for any $i, j,\ i \ne j$, then σ is a permutation on Z_d. When σ is a permutation on Z_d, we represent σ by a product of cyclic permutations such as $(\sigma_{k_1}\ \sigma_{k_2}\ \ldots \sigma_{k_{d-1}})$.

Next, we construct a mapping $\phi_{\boldsymbol{f},\sigma}$. For $\boldsymbol{f} = (f_1,\ f_2,\ f_3,\ \ldots,\ f_D,\ f')$, $\sigma = [\sigma_0\ \sigma_1\ \sigma_2 \ldots \sigma_d - 1]$ and $\boldsymbol{a} = (a_1,\ a_2,\ a_3,\ \ldots,\ a_D)$, $\phi_{\boldsymbol{f},\sigma}(\boldsymbol{a})$ is defined as follows:

$$\phi_{\boldsymbol{f},\sigma}(\boldsymbol{a}) = (a_2,\ a_3,\ \ldots,\ a_D,\ \boldsymbol{f}_\sigma(\boldsymbol{a})).$$

Lemma 2.1 *The mapping* $\phi_{\boldsymbol{f},\sigma}$ *is a bijection on* Z_d^D, *for* $\boldsymbol{f} = (f_1,\ f_2,\ f_3,\ \ldots,\ f_D,\ f')$, $\gcd(f_1, d) = 1$ *and a bijection* σ *on* Z_d.

PROOF We can only show that the mapping $\phi_{\boldsymbol{f},\sigma}$is an injection on Z_d^D. Let $\boldsymbol{a} = (a_1,\ a_2,\ a_3,\ \ldots,\ a_D)$ and $\boldsymbol{b} = (b_1,\ b_2,\ b_3, \ldots,\ b_D)$ in Z_d^D, and assume that $\phi_{\boldsymbol{f},\sigma}(\boldsymbol{a}) = \phi_{\boldsymbol{f},\sigma}(\boldsymbol{b})$. Then $a_i = b_i$, $(2 \le i \le D)$ and $\boldsymbol{f}_\sigma(\boldsymbol{a}) = \boldsymbol{f}_\sigma(\boldsymbol{b})$. Since σ is a bijection, we obtain $f_1a_1 + f_2a_2 + \cdots + f_Da_D + f' = f_1b_1 + f_2b_2 + \cdots + f_Db_D + f'$. Hence $f_1a_1 = f_1b_1$. Since $\gcd(f_1, d) = 1$, we obtain $a_1 = b_1$. ■

We construct a mapping $\varphi_{\boldsymbol{f},\sigma}$ from Z_d^D to Z_d^{D+1}. For $\boldsymbol{f} = (f_1,\ f_2,\ f_3,\ \ldots,\ f_D,\ f')$, $f_1 \ne 0$, $\sigma = [\sigma_0\ \sigma_1\ \sigma_2 \ldots \sigma_d - 1]$ and $\boldsymbol{a} = (a_1,\ a_2,\ a_3,\ \ldots,\ a_D)$, $\varphi_{\boldsymbol{f},\sigma}$ is defined as follows: $\varphi_{\boldsymbol{f},\sigma}(\boldsymbol{a}) = \langle \boldsymbol{a}, \boldsymbol{f}_\sigma(\boldsymbol{a})\rangle$. Let S be a subset of Z_d^{D+1} obtained by $\varphi_{\boldsymbol{f},\sigma}$. Then we can identify S with a subset of $A(B(d, D))$. If $\langle \boldsymbol{a}, \boldsymbol{f}_\sigma(\boldsymbol{a})\rangle \in S$ then $\boldsymbol{a}$ is adjacent to $\phi_{\boldsymbol{f},\sigma}(\boldsymbol{a})$. Hence S induces a subgraph of $B(d, D)$. So we call S an *arc induced subdigraph by* $\varphi_{\boldsymbol{f},\sigma}$ and $\varphi_{\boldsymbol{f},\sigma}$ an *arc induced mapping*, and S is denoted by $C_{\boldsymbol{f},\sigma}$. If $\varphi_{\boldsymbol{f},\sigma}$ is an injection, then $C_{\boldsymbol{f},\sigma}$ is a factor of $B(d, D)$ and components in $C_{\boldsymbol{f},\sigma}$ are directed cycles.

Lemma 2.2 *Let* $\boldsymbol{f} = (f_1,\ f_2,\ f_3,\ \ldots,\ f_D,\ f')$, $\gcd(f_1, d) = 1$ *and* σ *a bijection on* Z_d. *Then for any de Bruijn digraph* $B(d, D)$, $\varphi_{\boldsymbol{f},\sigma}$ *is an arc induced mapping and* $C_{\boldsymbol{f},\sigma}$ *is constituted of directed cycles.*

PROOF It is easy to see that $\varphi_{\boldsymbol{f},\sigma}$ is an arc induced mapping, from Lemma 2.1 and definition of $\varphi_{\boldsymbol{f},\sigma}$. Since $\boldsymbol{f} = (f_1,\ f_2,\ f_3,\ \ldots,\ f_D,\ f')$, $f_1 \ne 0$ and σ is a bijection on Z_d, $\phi_{\boldsymbol{f},\sigma}$ is a bijection on Z_d^D. Hence $\varphi_{\boldsymbol{f},\sigma}$ is an injection from Z_d^D to Z_d^{D+1}. Thus $C_{\boldsymbol{f},\sigma}$ is constituted of directed cycles. ■

We consider two special cases such that $B(d, D)$ is factorized by arc induced subdigraphs.

Theorem 2.3 *Let* $\boldsymbol{f} = (f_1,\ f_2,\ f_3, \ldots, f_D,\ f')$, $\gcd(f_1, d) = 1$ *and* $\sigma = (\sigma_0\ \sigma_1\ \sigma_2 \ldots \sigma_{d-1})$ *be a cyclic permutation of length* d *and* $C_{\boldsymbol{f},\sigma}$ *an arc induced subdigraph by* $\varphi_{\boldsymbol{f},\sigma}$.
Then $C_{\boldsymbol{f},\sigma} \oplus C_{\boldsymbol{f},\sigma^2} \oplus \cdots \oplus C_{\boldsymbol{f},\sigma^d} = B(d, D)$ *and* $C_{\boldsymbol{f},\sigma^i}$ $(1 \le i \le d)$ *is constituted of directed cycles.*

PROOF From Lemma 2.2, for an integer i between 1 and d, $C_{\boldsymbol{f},\sigma^i}$ is constructed by directed cycles and a factor of $B(d, D)$, where $C_{\boldsymbol{f},\sigma^1} =$

$C_{\boldsymbol{f},\sigma}$. Furthermore $|E(C_{\boldsymbol{f},\sigma^i})| = d^D$. Let i, j be distinct integers between 1 and d. We show $E(C_{\boldsymbol{f},\sigma^i}) \cap E(C_{\boldsymbol{f},\sigma^j}) = \emptyset$. Suppose there is an arc that is in $E(C_{\boldsymbol{f},\sigma^i}) \cap E(C_{\boldsymbol{f},\sigma^j})$. Then there is a vector $\boldsymbol{a} = (a_1,\ a_2,\ a_3,\ \ldots,\ a_D)$ such that $\varphi_{\boldsymbol{f},\sigma^i}(\boldsymbol{a}) = \varphi_{\boldsymbol{f},\sigma^j}(\boldsymbol{a})$. Thus we obtain $\phi_{\boldsymbol{f},\sigma^i}(\boldsymbol{a}) = \phi_{\boldsymbol{f},\sigma^j}(\boldsymbol{a})$. Hence this contradicts to a cyclic permutation σ of length d. ■

Theorem 2.4 *Let* $\boldsymbol{f}_i = (f_1,\ f_2,\ f_3,\ \ldots,\ f_D,\ i)$, $\gcd(f_1, d) = 1$, $(0 \le i \le d-1)$ *,* σ *a bijection on* Z_d *and* $C_{\boldsymbol{f}_i,\sigma}$ *an arc induced subdigraph by* $\varphi_{\boldsymbol{f}_i,\sigma}$. *Then* $C_{\boldsymbol{f}_0,\sigma} \oplus C_{\boldsymbol{f}_1,\sigma} \oplus \cdots \oplus C_{\boldsymbol{f}_{d-1},\sigma} = B(d, D)$, *and* $C_{\boldsymbol{f}_i,\sigma}$ *(*$1 \le i \le$*) is constituted of directed cycles.* ■

We call a factor F of G a *cycle component factor* of G, if all components of F are cycles(directed cycles). We consider three special cases with respect to $\boldsymbol{f}$ and σ.

Theorem 2.5 *Let* $\boldsymbol{f} = (1,\ 0,\ 0,\ \ldots,\ 0,\ 0) \in Z_d^D$ *and* $\sigma = \begin{bmatrix} 0 & 1 & 2 & 3 & \ldots & d-1 \\ 0 & 1 & 2 & 3 & \ldots & d-1 \end{bmatrix}$. *Then* $C_{\boldsymbol{f},\sigma}$ *is a factor of* $B(d, D)$ *and the size of component in* $C_{\boldsymbol{f},\sigma}$ *is a divisor of* D. ■

In Fig. 3, bold and dashed arcs are $\langle \boldsymbol{a}, \boldsymbol{f}_\sigma \rangle$ where $\boldsymbol{f} = (1, 0, 0, 0, 0)$, $\sigma = [0\ 1]$ and $\boldsymbol{a} \in V(B(2, 4))$.

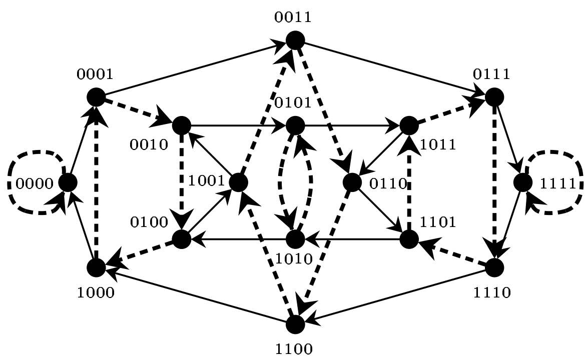

Fig. 3: $C_{\boldsymbol{f},\sigma}$ in $B(2, 4)$ where $\boldsymbol{f} = (1, 0, 0, 0, 0)$ and $\sigma = [0\ 1]$.

Theorem 2.6 *Let* i *and* j *be any integers. Let* $\boldsymbol{f} = (1\ 1\ 1\ \ldots\ 1\ i) \in Z_d^{D+1}$ *and*
$$\sigma = \begin{bmatrix} 0 & 1 & \ldots & k & \ldots & d-1 \\ j & j-1 & \ldots & j-k & \ldots & j-(d-1) \end{bmatrix},$$
where $j-k$ $(1 \le k \le d-1)$ *is taken under modulus* d. *Then* $C_{\boldsymbol{f},\sigma}$ *is a factor of* $B(d, D)$ *and the size of component in* $C_{\boldsymbol{f},\sigma}$ *is a divisor of* $D+1$.

PROOF $C_{\boldsymbol{f},\sigma}$ is a cycle component factor of $B(d, D)$ from Lemma 2.2. We can only consider the following case : $\boldsymbol{f} = (1\ 1\ 1\ \ldots\ 1\ 0) \in Z_d^{2^n}$ and
$$\sigma = \begin{bmatrix} 0 & 1 & \ldots & k & \ldots & d-1 \\ j & j-1 & \ldots & j-k & \ldots & j-(d-1) \end{bmatrix}$$
, where $j-k$ $(1 \le k \le d-1)$ is taken under modulus d. We show $\boldsymbol{f}_\sigma(\phi_{\boldsymbol{f},\sigma}(\boldsymbol{a})) = a_1$ for any $\boldsymbol{a} = \{a_1\ a_2\ a_3\ \ldots\ a_D\} \in Z_d^D$. From the definition, $\phi_{\boldsymbol{f},\sigma}(\boldsymbol{a}) = (a_2\ a_3\ \ldots\ a_D \boldsymbol{f}_\sigma(\boldsymbol{a}))$. From the definitions of $\boldsymbol{f}$ and σ,
$$\boldsymbol{f}_\sigma(\boldsymbol{a}) \equiv j - (a_1 + a_2 + \cdots + a_D) \pmod{d}. \quad (1)$$
Then $\boldsymbol{f}_\sigma(\phi_{\boldsymbol{f},\sigma}(\boldsymbol{a})) \equiv j-(a_2+a_3+\cdots+a_D+\boldsymbol{f}_\sigma(\boldsymbol{a}))$ (mod d). From (1), we obtain $\boldsymbol{f}_\sigma(\phi_{\boldsymbol{f},\sigma}(\boldsymbol{a})) = a_1$ ■

Theorem 2.7 *Let* i *be any integer. Let* $\boldsymbol{f} = (1\ 0\ 1\ 0\ 1\ 0 \ldots\ 1\ 0\ 1\ 0\ i) \in Z_d^{2D+1}$ *and*
$$\sigma = \begin{bmatrix} 0 & 1 & \ldots & k & \ldots & d-1 \\ j & j-1 & \ldots & j-k & \ldots & d-1 \end{bmatrix}$$
,where $j-k$ $(1 \le k \le d-1)$ *is taken under modulus* d. *Then* $C_{\boldsymbol{f},\sigma}$ *is a cycle component factor of* $B(d, 2D)$ *and the size of a component in* $C_{\boldsymbol{f},\sigma}$ *is a divisor of* $2D+2$.

PROOF We can show this statement by similar way to the proof of Theorem 2.6. ■

3 Finding independent set on $UB(d, D)$

The *line graph* $L(G)$ of a graph G is the graph whose vertices can be put in one-to-one correspondence with the edges of G in such a way that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. It is well known that $\beta_1(G) = \beta_0(L(G))$ for any undirected graph with no loops and no multiple edges [4]. If G is a multigraph having loops and multiple edges, then $\beta_1(G) \le \beta_0(L(G))$. Similarly, for a digraph G with no loops and no multiple arcs, $\beta_1(G) \le \beta_0(U(L(G)))$. Furthermore for a multidigraph G having loops and multiple edges, we could say that $\beta_1(G) \le \beta_0(U(L(G)))$. Now we introduce the notion of *weak arc independence*. A set $T \in A(G)$ is a *weak arc independent set* of G, if no two distinct arcs in T are adjacent. If T is a weakly arc independent set, then T could contain loops. The *weak matching number* $w\beta_1(G)$ of G is the maximum cardinality of weakly independent arc set in G. Then for a digraph and multidigraph G, we can see that $\beta_1(G) \leq w\beta_1(G)$ and $w\beta_1(G) \leq \beta_0(U(L(G)))$. Furthermore a weakly arc independent set of digraph G corresponds to the independent set of $U(L(G))$. Thus maximal weakly arc independent set T for a digraph and multidigraph has one to one correspondence to the cover of $U(L(G))$. Therefore we obtain $w\beta_1(B(d, D-1)) \leq \beta_0(UB(d,D))$ for the de Bruijn graph $UB(d,D)$ $(2 \leq D)$ and the maximal weakly arc independent set for $B(d, D-1)$ corresponds to the maximal independent set of $UB(d,D)$.

A *colouring* of a graph(digraph) G is an assignment of colours to the vertices of G, one colour to each vertex, so that adjacent vertices are assigned different colours. A colouring in which k colours are used is a k*-colouring*. The minimum integer k for which a graph G is k- colourable is called the *chromatic number* of G, and is denoted by $\chi(G)$. An assignment of colours to the edges(arcs) of a nonempty graph(digraph) G so that adjacent edges(arcs) are coloured differently is an *edge colouring*(*arc colouring*) of G (a k*-edge colouring*(k*-arc colouring*) if k colours are used). The minimum k for which a graph G is k-edge(k-arc) colourable is its *chromatic index* and is denoted by $\chi_1(G)$. It is clear that $\chi_1(G) = \chi(L(G))$. For an odd cycle C, $\chi_1(C) = 3$, and if C has the even length, then $\chi_1(C) = 2$.

We construct cover of $UB(d,D)$ using the following algorithm. We can see that this algorithm takes linear time for the number of vertices.

Algorithm 1

input	:	de Bruijn graph $UB(d,D)$, i.e. two integer d and D
output	:	an independent set of $UB(d,D)$
k, t	:	integers
n	:	number of component in cycle component factor $C_{\boldsymbol{f},\sigma}$
EC_k	:	the arc set of component C_k of $C_{\boldsymbol{f},\sigma}$
$k(m)$	:	number of arcs in the component EC_k
$a_{k(t)}$	:	the arc in the component EC_k
A_R	:	the arc set whose arc is coloured red
A_B	:	the arc set whose arc is coloured blue
A_Y	:	the arc set whose arc is coloured yellow
V_C	:	the vertex set that corresponds edge set of $C_{\boldsymbol{f},\sigma}$ of $UB(d,D)$
V_R	:	the vertex set that corresponds A_R of $UB(d,D)$
$V_{UB(d,D)}$	:	the vertex set of $UB(d,D)$

generate a cycle component factor $C_{\boldsymbol{f},\sigma}$ of $B(d, D-1)$ using d and D;

```
do{
    k := k + 1;
    t := 0;
    if(k(m) := 1){
        t := t + 1;
        colour the arc a_{k(t)} coloured red;
        A_R := A_R ∪ {a_{k(t)}};
    }
    else if(k(m) is even){
        do{
            t := t + 1;
            if(t is odd){
                colour a_{k(t)} ∈ EC_k red;
                A_R := A_R ∪ {a_{k(t)}};
            }
            else{
                colour a_{k(t)} ∈ EC_k blue;
                A_B := A_B ∪ {a_{k(t)}};
            };
        }while(t < k(m));
    }
    else{
        do{
            if(t = 0){
                colour a_{k(t)} ∈ EC_k yellow;
                A_Y := A_Y ∪ {a_{k(t)}};
            }
            else if(t is odd){
                colour a_{k(t)} ∈ EC_k red;
                A_R := A_R ∪ {a_{k(t)}};
            }
            else{
                colour a_{k(t)} ∈ EC_k blue;
                A_B := A_B ∪ {a_{k(t)}};
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            };
            t := t + 1;
            }while(t < k(m));
      };
}while(k < n);
return (V_{UB(d,D)} \ (V_C ∪ I(V_R) ∪ O(V_R))) ∪ V_R;
```

Theorem 3.1 *We can obtain a cover of $UB(d, D)$ by using by Algorithm* 1. ■

we can obtain a cover whose cardinality is 3 for $B(2,3)$. Furthermore this cover is a maximum independent set of $B(2,3)$. We would like to obtain a cover that contains as many vertices as possible. Then we can obtain covers for special cases using Algorithm 1 and applying Theorem 2.5, 2.6 and 2.7.

Corollary 3.2 *Let $\boldsymbol{f} \in Z_d^{2^n}$ and σ that is a bijection on Z_d be the similar one to those in Theorem 3. We can obtain a cover of $B(d, 2^n+1)$ by Algorithm 1 and this cover contains at least $\dfrac{d^{2^n}+d}{2}$ vertices.* ■

Corollary 3.3 *Let $\boldsymbol{f} \in Z_d^{2^n-1}$ and σ that is a bijection on Z_d be the same one in Theorem 2.6. We can obtain a cover of $B(d, 2^n)$ by Algorithm 1 and this cover contains at least $\dfrac{d^{2^n-1}+d}{2}$ vertices.* ■

Corollary 3.4 *Let $\boldsymbol{f} \in Z_d^{2^n-2}$ and σ that is a bijection on Z_d be the same one in Theorem 2.7. We can obtain a cover of $B(d, 2^n-1)$ by Algorithm 1 and this cover contains at least $\dfrac{d^{2^n-2}+d}{2}$ vertices.* ■

Reference

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