Controller Synthesis of Discrete Time Fuzzy Systems Based on Piecewise Lyapunov Functions

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Abstract: - This paper presents a controller synthesis method for the discrete time fuzzy systems based on a piecewise Lyapunov function. The basic idea of the proposed approach is to construct controllers for the fuzzy systems in such a way that a piecewise Lyapunov function can be used to establish the global stability of the resulting closed loop fuzzy control systems. It is shown that the control laws can be obtained by solving a set of Linear Matrix Inequalities (LMI) that is numerically feasible with commercially available software. Two examples are given to demonstrate the advantage and the applicability of the proposed method.

Key-Words: - Discrete time, Fuzzy systems, Linear matrix inequality, Piecewise Lyapunov function, Controller synthesis.

1 Introduction

Fuzzy logical control (FLC) has recently proved to be a successful control approach for certain complex nonlinear systems, see [1–5] for example. However, conventional fuzzy control system [1] has proved extremely difficult to be analyzed and designed. The reason for this is believed to be due to the fact that no mathematical model is available from the conventional fuzzy system.

Recently, there have appeared a number of stability analysis and controller synthesis results in fuzzy control literature, where the Takagi-Sugeno's fuzzy models [6] are used, see references in [7-14]. The stability of the overall fuzzy system is determined by checking a Lyapunov function or a set of Linear Matrix Inequalities (LMI). It is required that a common positive definite matrix P can be found to satisfy the Lyapunov function or the set of LMIs. However, this is a difficult problem to solve since such a matrix might not exist in many cases, especially for highly nonlinear complex systems. Most recently, a stability result of fuzzy systems using a piecewise quadratic Lyapunov function has been reported [15]. It is also demonstrated in the paper that the piecewise Lyapunov function is a much richer class of Lyapunov function candidates than the common Lyapunov function candidates and thus it is able to deal with a larger class of fuzzy systems. Further references to piecewise Lyapunov functions can be found in [17-20].

During the last few years, we have proposed a number of new methods for the systematic analysis and design of fuzzy logic controllers based on a so-called fuzzy dynamic model which is similar to the Takagi-Sugeno's model [7-11]. These methods include designs based on a nominal model, a common Lyapunov function and a piecewise Lyapunov function. However, for the methods based on the piecewise Lyapunov function, certain restrictive boundary conditions have to be imposed.

Motivated from the results of continuous time piecewise Lyapunov functions in [15], we developed a new stability theorem for discrete time fuzzy systems based on a piecewise Lyapunov function in [16]. In this paper, we extend the same idea to a new controller synthesis method for same class of fuzzy systems. It should be noted that with this kind of piecewise Lyapunov function, the restrictive boundary condition existing in our previous analysis can be removed and global stability of the closed loop fuzzy control system can be easily established. Moreover, the controller synthesis procedure is to solve a set of LMIs that is numerically feasible with commercially available software.

The rest of the paper is organized as follows. Section 2 introduces the discrete time fuzzy system. Section 3 defines the piecewise Lyapunov function candidate, introduces the stability theorem and then presents a new controller synthesis method for such systems. Two examples are given in same section to demonstrate the advantage and the applicability of the proposed approach. Finally, conclusions are given in Section 4.

2 Problem Formulation

The fuzzy dynamic model proposed in [7-11] can be used to represent a complex discrete-time system with both fuzzy inference rules and local analytic linear models,

$$\begin{aligned} R^l: \quad & \text{IF} \quad x_1 \text{ is } F_1^l \text{ AND} \ldots x_n \text{ is } F_n^l \\ & \text{THEN} \quad x(t+1) = A_l x(t) + B_l u(t), \\ & \qquad l = 1,2,\ldots,m \end{aligned} \tag{1}$$

where R^l denotes the *l*-th fuzzy inference rule, *m* the number of inference rules, F_j^l *(j=1,2,...,n)* are fuzzy sets, $x(t) \in \Re^n$ the system state variables, $u(t) \in \Re^p$ the system input variables, and (A_l, B_l) is the *l-th* local model of the fuzzy system (1).

Assumption 2.1: For the controller synthesis in the following sections it is assumed that the fuzzy system (1) is locally controllable, that is, all the pairs (A_l, B_l), $l = 1,2,\ldots,m$, are controllable.

Let $m_l(x(t))$ be the normalized membership function of the inferred fuzzy set F^l where $F^l = \prod_{j=1}^{n} F_j^l$, and is defined as

$$0 \le m_l(x(t)) \le 1 \text{ and } \sum_{l=1}^{m} m_l = 1. \tag{2}$$

By using a center-average defuzzifer, product inference and singleton fuzzifier [7-11], the discrete time fuzzy system (1) can be expressed by the following global model

$$x(t+1) = \sum_{l=1}^{m} m_l(x(t)) \cdot \{A_l x(t) + B_l u(t)\}, \; x(t) \in \Re^n \tag{3}$$

Define *L* as the set of subspace indexes. Since the rules of the fuzzy system (1) induce a polyhedral partition $\{S_i\}_{i \in L} \subseteq \Re^n$ of the state space, the fuzzy system (3) can be viewed as a number of subsystems in a set of individual subspaces, which consist of crisp (operating), and fuzzy (interpolation) subspaces.

The crisp subspace is defined as the subspace where $m_l(x) = 1$ for some *l*, and all other membership functions evaluate to zero. The system dynamics of crisp subspace is given by *l-th* local model of the fuzzy system (1). On the other hand, the fuzzy subspace is defined as the subspace where $0 < m_l(x) < 1$ and the system dynamics is given by a convex combination of several linear systems.

In the extreme case where all the subspaces of a fuzzy system are crisp, that is, $m_l(x(t)) = 1$ for some *l* and all other membership functions are equal to zero, then fuzzy system (3) becomes a piecewise linear system, $x(t+1) = A_l x(t) + B_l u(t)$. However, in many cases, the membership function, $m_l(x(t))$ for some *l*, could be between 0 and 1.

In our previous attempts [7-11], we treated the fuzzy dynamic systems in the fuzzy subspace in terms of uncertainties and used upper bound approximation of those uncertainties to perform stability analysis. However, in this paper, we will follow the idea of [15] to write the fuzzy system (3) in the fuzzy subspaces as a convex combination of linear systems

$$x(t+1) = \sum_{k \in K(i)} m_k(x(t)) \{A_k x(t) + B_k u(t)\}, \; x(t) \in S_i \tag{4}$$

with $0 \le m_k(x) \le 1$, $\sum_{k \in K(i)} m_k(x) = 1$. For each subspace S_i, the set $K(i)$ contains the indexes for the system matrices used in the interpolation within that subspace. For crisp subspace $K(i)$ contains a single element.

Assumption 2.2: We assume that given any initial condition $x(0) = x_0$, the global model (4) has a unique solution for all $t \ge 0$.

Assumption 2.3: We also assume that when the state of the system transits from the subspace S_i to S_j at the time *t*, the dynamics of the system is governed by the dynamics of the local model of S_i at that time.

For future use, we also define a set Ω that represents all possible transitions from one subspace to another, that is,

$$\Omega := \{i, j \mid x(t) \in S_i, x(t+1) \in S_j, i \ne j\} \tag{5}$$

Remark 2.1: Due to the discrete nature of the system, it is noted that Ω as in (5) could include transitions occurred between non-adjacent subspaces in one step.

Note that in comparison with (3), the fuzzy system (4) is described in each subspace.

In this section, we will first present the stability

development of our controller synthesis method.

fuzzy system can be analysed by piecewise quadratic

discrete time case. The general idea of our approach

(4). Then we can use the piecewise Lyapunov function to check t
system.

Due to the fact that the state in discrete time fuzzy systems will most likely jump between
-adjacent subspaces, the structural
F's in [15], cannot be
haracterize the state transition from one subspace to another as dealt with in the case of

may not be helpful to construct a piecewise Lyapunov function that is continuous across
ete time fuzzy systems to analyse stability of the system as in [15] for the

also be unnecessary to require the piecewise Lyapunov function to be continuous across
ystems since
the state of such systems may never pass through the

As for the Lyapunov function candidate, we

$$V(t) = {}^{t}P_i x \qquad x(t) \in {}_{i}\,,\ i \tag{6}$$

Then we are read
result of the paper [16].

Theorem 3.1 [16]: Consider the global discrete time fuzzy system (4) with $u \equiv 0$. If there exist symmetric matrices P_i, P_j, $i, j \in L$ and the following LMIs are satisfied,

$$0 < P_i\,, \qquad i \in L \tag{7}$$

$$A_k^T P_i A_k - P_i < 0\,,\ i \in L\,,\ k \in K(i) \tag{8}$$

$$A_k^T P_j A_k - P_i < 0\,,\ i, j \in \Omega \cap L\,,\ k \in K(i) \tag{9}$$

then the discrete time fuzzy system is globally exponentially stable, that is, *x(t)* tends to the origin exponentially for every trajectory in the state space.

Proof: See [16].

The above conditions are linear matrix inequalities in the variables P_i, P_j. A solution to those inequalities ensures *V(t)* defined in (6) to be a piecewise Lyapunov function for the system. The LMI (8) guarantees that the function decreases along all system trajectories within each subspace. The LMI (9) guarantees that the function is decreasing when the state transits from one subspace to another.

Remark 3.1: For each fuzzy subspace, we are seeking for a "common" piecewise quadratic Lyapunov function which can satisfy all the partial influencing system state matrices, and which decrease in time within or between the subspace.

Remark 3.2: Due to the discrete nature of the system, it is noted that transitions could occur between non-adjacent subspaces in one step. Thus, every subspace pair, (S_i, S_j) as defined by (5), has to be computed in (9).

Remark 3.3: It is noted that when the state of the system does transit across the boundaries, that is, $x(t) \in S_i \cap S_j$ for some *t*, the result in Theorem 3.1 still holds since the case can be covered by considering the transition from the subspace S_i to S_j at the time *t* or $t+1$.

Remark 3.4: Theorem 3.1 is only a sufficient condition for system stability. Thus, the discrete time fuzzy system may still be stable even if the piecewise Lyapunov function (6) can not be identified from the above inequalities. Shall Theorem 3.1 fail to generate solutions, one may refine the partition in order to increase the flexibility of the Lyapunov function candidate and try anew [15].

Remark 3.5: The stability test of the discrete time fuzzy system in (7)-(9) can be easily facilitated by a commercially available software package Matlab LMI toolbox [21-22].

Now, we will address the controller synthesis problem for the discrete time fuzzy systems (1). The proposed state feedback controller synthesis is based on linear system(s) defined in each subspace. For the stabilization of the fuzzy system (1), we consider the following piecewise discrete time controller as

$$u(t) = \hat{K}_i x(t)\,, \quad x(t) \in S_i\,,\ i \in L \tag{10}$$

So the closed loop discrete time fuzzy control system is

$$x(t+1) = \sum\nolimits_{k \in K(i)} m_k(x(t)) A_{ck}\, x(t)\,,\ x(t) \in S_i \tag{11}$$

where

$$A_{ck} = A_k + B_k \hat{K}_i\,,\ k \in K(i)\,,\ i \in L \tag{12}$$

Then we have the following results.

Theorem 3.2: The fuzzy control system (11) is globally exponentially stable to the origin, if there exist matrices Y_i, $i \in L$, symmetric matrices $R_i, R_j, i, j \in L$, such that the following LMIs are satisfied,

$$0 < R_i, \qquad i \in L \tag{13}$$

$$\begin{bmatrix} -R_i & (A_k R_i + B_k Y_i)^t \\ (A_k R_i + B_k Y_i) & -R_i \end{bmatrix} < 0, \quad i \in L,\ k \in K(i) \tag{14}$$

$$\begin{bmatrix} -R_i & (A_k R_i + B_k Y_i)^t \\ (A_k R_i + B_k Y_i) & -R_j \end{bmatrix} < 0, \quad i, j \in \Omega \cap L,\ k \in K(i). \tag{15}$$

Moreover, the control law for each subspace is given by

$$\hat{K}_i = Y_i R_i^{-1},\ i \in L \tag{16}$$

Proof: Based on the result in Theorem 3.1 and its proof, we learn that the fuzzy control system (11) is globally exponentially stable if there exist symmetric positive definite matrices P_i, P_j satisfying the following inequalities,

$$A_{ck}^T P_i A_{ck} - P_i < 0, \qquad i \in L,\ k \in K(i) \tag{17}$$

$$A_{ck}^T P_j A_{ck} - P_i < 0, \qquad i, j \in \Omega \cap L, k \in K(i). \tag{18}$$

We will first show that the inequality (14) is equivalent to (17). Using Schur's complement, (14) is equivalent to

$$-R_i < 0,$$

$$-R_i + (A_k R_i + B_k Y_i)^t R_i^{-1} (A_k R_i + B_k Y_i) < 0 \tag{19}$$

Since $Y_i = \hat{K}_i R_i$, $R_i = R_i^t$, and $A_{ck} = A_k + B_k \hat{K}_i$, (19) becomes

$$-R_i + (A_{ck} R_i)^t R_i^{-1} (A_{ck} R_i) < 0 \tag{20}$$

Multiply (20) with P_i both sides of each term, with the fact $R_i = P_i^{-1}$, we have

$$A_{ck}^T P_i A_{ck} - P_i < 0.$$

Thus, we have shown that the inequality (14) is equivalent to (17). Following the above procedure, it can also be shown that the inequality (15) is equivalent to (18). Therefore, it can be concluded that the closed loop fuzzy control system is globally exponentially stable and thus the proof is completed. $\nabla\nabla$

Example 1: Consider the discrete time fuzzy system that switches between 3 rules,

R^1: IF x_1 is about negative
THEN $x(t+1) = A_1 x(t) + B_1 u(t)$

R^2: IF x_1 is about zero

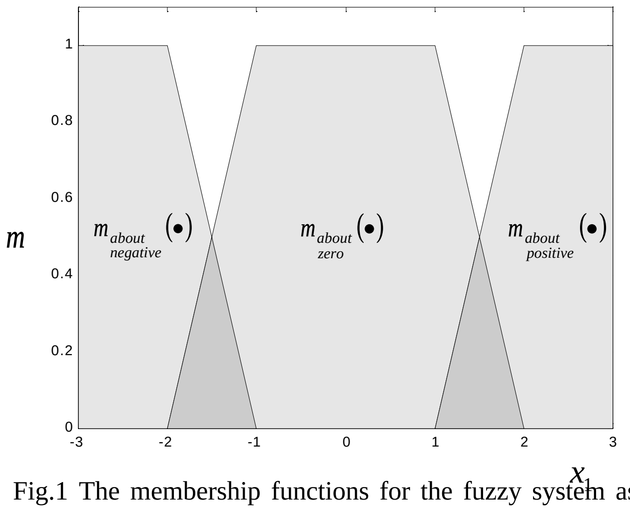

Fig.1 The membership functions for the fuzzy system as in Example 1.

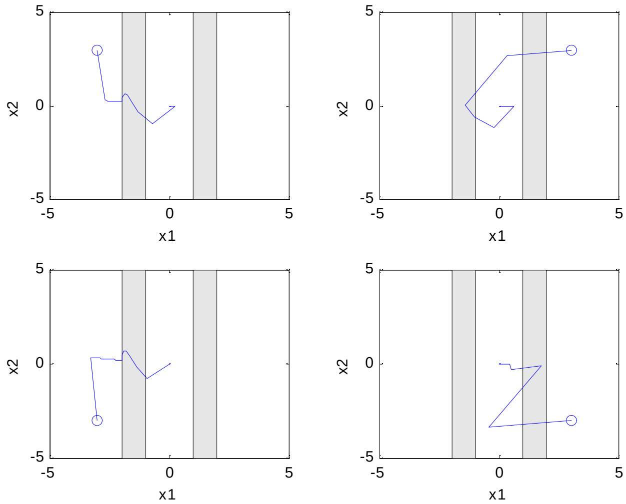

Fig.2. Trajectory of closed loop fuzzy control system with different initial conditions, $[-3\ \ 3]^t$, $[3\ \ 3]^t$, $[-3\ \ -3]^t$ and $[3\ \ -3]^t$.

THEN $x(t+1) = A_2 x(t) + B_2 u(t)$

R^3: IF x_1 is about positive
THEN $x(t+1) = A_3 x(t) + B_3 u(t)$

where the membership function for "about negative", "about zero", and "about positive" is defined as in Fig.1. The system matrices are given by

$$A_1 = \begin{bmatrix} 1 & 0.1 \\ -0.5 & 1 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.5 & -0.6 \\ 0.6 & 0.5 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 0.5 \\ -0.1 & 1 \end{bmatrix},\ B_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

It is noted that the open loop fuzzy system is unstable, from Theorem 3.1 and simulations, and that there is no solution to the common quadratic Lyapunov function approach. That is, if a common positive definite matrix P is used in Theorem 3.2, then there is no solution to those LMIs. However, if using the piecewise Lyapunov function approach proposed in this paper, we can obtain the following feasible solutions to those 40 LMIs.

$$R_1=\begin{bmatrix}3.4534 & -0.7838\\ -0.7838 & 8.3368\end{bmatrix},\ R_2=\begin{bmatrix}3.4025 & -0.3150\\ -0.3150 & 5.2240\end{bmatrix},$$

$$R_3=\begin{bmatrix}6.2335 & 1.5148\\ 1.5148 & 6.3328\end{bmatrix},\quad R_4=\begin{bmatrix}6.3518 & 0.8862\\ 0.8862 & 4.7505\end{bmatrix},$$

$$R_5=\begin{bmatrix}7.6188 & 0.9519\\ 0.9519 & 4.7547\end{bmatrix},$$

$$K_1=[0.3913 \quad -1.0108],\quad K_2=[0.3336 \quad -0.1431],$$

$$K_3=[-0.6214 \quad -0.4744],\ K_4=[-0.7713 \quad -0.3978],$$

$$K_5=[-1.0128 \quad -0.3710].$$

It thus follows from Theorem 3.2 that the stability of the closed loop fuzzy control system is guaranteed. Simulation results of four different initial conditions are shown in Fig.2. which illustrate the stability of the system.

This example clearly demonstrates the advantage of the piecewise Lyapunov function approach to the common Lyapunov function approach.

Example 2: We now apply the above synthesis technique to backing up control of a computer simulated truck trailer. We use the following truck-trailer model formulated in [5]:

$$x_1(t+1)=\left(1-v\cdot\tilde{t}/L\right)x_1(t)+v\cdot\tilde{t}/l\cdot u(t)$$

$$x_2(t+1)=x_2(t)+v\cdot\tilde{t}/L\cdot x_1(t)$$

$$x_3(t+1)=x_3(t)+v\cdot\tilde{t}\cdot\sin\left[x_2(t)+v\cdot\tilde{t}/2L\cdot x_1(t)\right].$$

The following fuzzy system [5] is used to design a fuzzy controller:

R^1: IF $z(t)=x_2(t)+\frac{v\cdot\tilde{t}}{2L}\cdot x_1(t)$ is about zero

THEN $x(t+1)=A_1x(t)+B_1u(t)$

R^2: IF $z(t)=x_2(t)+\frac{v\cdot\tilde{t}}{2L}\cdot x_1(t)$ is about p or -p

THEN $x(t+1)=A_2x(t)+B_2u(t)$

where

$$A_1=\begin{bmatrix}1-\frac{v\cdot\tilde{t}}{L} & 0 & 0\\ \frac{v\cdot\tilde{t}}{L} & 1 & 0\\ \frac{v^2\cdot\tilde{t}^2}{2L} & v\cdot\tilde{t} & 1\end{bmatrix},\qquad B_1=\begin{bmatrix}\frac{v\cdot\tilde{t}}{l}\\ 0\\ 0\end{bmatrix},$$

$$A_2=\begin{bmatrix}1-\frac{v\cdot\tilde{t}}{L} & 0 & 0\\ \frac{v\cdot\tilde{t}}{L} & 1 & 0\\ \frac{d\cdot v^2\cdot\tilde{t}^2}{2L} & d\cdot v\cdot\tilde{t} & 1\end{bmatrix},\qquad B_2=\begin{bmatrix}\frac{v\cdot\tilde{t}}{l}\\ 0\\ 0\end{bmatrix},$$

$l=2.8$, $L=5.5$, $v=-1.0$, $\tilde{t}=2.0$, $d=0.01/p$.

We use the trapezoidal membership functions as defined in Fig.3 for "about zero", "about p " and "about -p ".

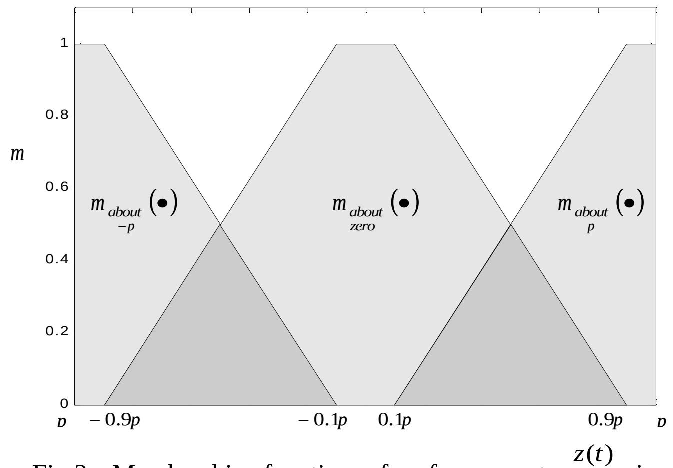

Fig.3. Membership functions for fuzzy system as in Example 2.

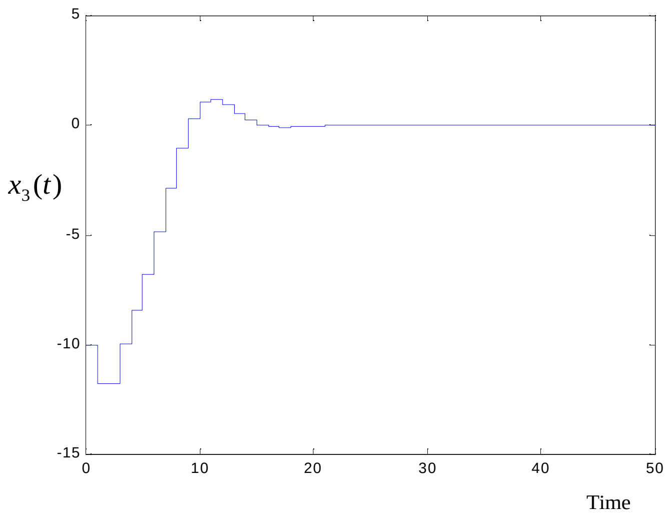

Fig.4. Closed loop truck-trailer position response, with initial condition of $[0.5p \quad 0.75p \quad -10]^t$.

Using Theorem 3.2, we obtain the following feasible solutions,

$$R_1=R_5=\begin{bmatrix}16.7163 & 6.0803 & -0.0746\\ 6.0803 & 4.7145 & 2.8193\\ -0.0746 & 2.8193 & 14.4552\end{bmatrix},$$

$$R_2=R_4=\begin{bmatrix}16.7773 & 5.4925 & -0.0946\\ 5.4925 & 3.1152 & 2.8243\\ -0.0946 & 2.8243 & 14.4628\end{bmatrix},$$

$$R_3=\begin{bmatrix}17.1031 & 5.4607 & -0.8241\\ 5.4607 & 3.4991 & 4.1990\\ -0.8241 & 4.1990 & 20.0265\end{bmatrix},$$

$$K_1=K_5=[2.9442 \quad -2.8490 \quad 0.5645],$$

$$K_2=K_4=[3.0813 \quad -3.5617 \quad 0.7045],$$

$$K_3=[3.1362 \quad -3.9476 \quad 0.5612].$$

Thus the closed loop truck-trailer fuzzy control system is stable in Lyapunov sense. When the initial position of the truck-trailer is $\left[\frac{p}{2} \quad \frac{3p}{4} \quad -10\right]^t$, Fig.4

shows the position response of the system.

4 Conclusion

In this paper, a new method is developed to synthesize controllers of discrete time fuzzy systems based on a piecewise Lyapunov function. It is shown that the controllers can be obtained by solving a set of LMIs. Two examples are also presented to demonstrate the advantage and applicability of the proposed approach.

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