Algebraic Condition for Integrable Numerical Algorithms

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Abstract: - Algebraic Numerical Algorithm (ANA), especially Algebraic Finite Difference Equation (AFDCE) are treated by algebraic geometrically. By coherent sheaf and proper morphism condition [1], we can treat integrability of ANA and AFDCE. In this work we treated integrable properties of Linear AFDCE and Durand -Kerner -Aberth method from this condition as examples.

Key-Words: - Algebraic finite difference equation, Coherent sheaf, Analytic space, Integrability

1 Introduction

Criteria for integrability of discrete dynamical system, especially for the integrability of non-linear finite difference equation are proposed recently [2-3]. Using these criteria we can find out new non-linear AFDCEs that are candidates of integrable AFDCEs (IAFDCEs). These criteria are called Singularity Confinement and Algebraic Entropy criteria, respectively. Though we can get IAFDCEs by these criteria, we have not yet had clear theoretical background for these criteria.

On the other hand, algebraic treatment of discrete evolutional equations in control system was attempted [4]. In this approach, concepts of coherent sheaves were introduced to treat discontinuous dynamical systems properly.

From these works, GAGA principle was introduced into AFDCE [1, 5, 7, 8, 9, 10]. Though the principle is abstractive for practical application, we have to translate it into proper statement using orthodox numerical notation. The next few sections are prepared for this purpose. In these sections, we will introduce open coverings by Zariski topology, and sheaves and coherent sheaves by ideals from AFDCE. By these preparations we can define IAFDCEs as algebraic evolutional equations of which solution functions are holomorphic functions in analytic space by several complex variables except for appropriate singularities. In this work we introduce Stein space [11] as analytic space and proper space as algebraic space for AFDCE. GAGA principle connects analytic space and algebraic space. From this point, AFDCEs are functional equations in analytic space. Simple and important algebraic space which corresponds to analytic (Stein) space is projective space. Therefore we can confirm some validity of singularity confinement criterion. Moreover we can understand background of algebraic entropy criterion from this approach.

2 Coherent sheaves by AFDCE in affine space

Algebraic translations of AFDCE are shown in this section. Consider following simple 2-step algebraic finite difference equation as an example,

(1) $F(f_{n-1}, f_n, f_{n+1}) = 0$,

here n is integer and $F(f_{n-1}, f_n, f_{n+1}) \in C[f_{n-1}, f_n, f_{n+1}]$. $C[f_{n-1}, f_n, f_{n+1}]$ is polynomial function by $\{f_{n-1}, f_n, f_{n+1}\}$ with complex coefficient C. We write $C_n=C[f_{n-1}, f_n, f_{n+1}]$ and $F_n = F(f_{n-1}, f_n, f_{n+1})$. Here $f_j=f(z_j)$, $z_j\in C$, and j means order of the sequence of points $\{\ldots, z_{j-1}, z_j, z_{j+1}, z_{j+2}, \ldots\}$. Then we can regard (1) as functional equation of $f(z)$. It is known C_n is Noetherian (consists of finite number of ideals).

Consider localization by treating F_n as ideal in C_n. If F_n is irreducible polynomial (prime ideal) in C_n , then $C_n \backslash F_n$ becomes multiplicative set in C_n. We put $S_n=C_n \backslash F_n$ and $A_n= S_n^{-1}C_n$. Then A_n and F_nA_n become local ring and maximal ideal. (A_n,F_nA_n) is localization and A_n/F_nA_n is function field. We define $X_n=SpecA_n$ and X_n as all prime ideal of A_n. We call X_n as affine scheme at n.

We can consider morphism $j_j : X_j \to X_{j+1}$. This is also a kind of map or connection between X_j and X_{j+1} that should satisfy some condition for integrability.

Discrete analogy for sheaves of modules in AFDCE to usual affine scheme can be obtained as,

(i) Assume every F_j corresponds prime ideals. If F_j isn't prime ideal, we decompose it to prime ideals first.

(ii) Make A_n from F_j. Then A_n is Noetherian locally at least, because Ideals of A_n give sub-algebra of $C[f_{n-1}, f_n, f_{n+1}]$. Clearly $C[f_{n-1}, f_n, f_{n+1}]$ is Noetherian by Hilbert's basis theorem , therefore each A_n is Noetherian. A_n becomes O_{Fn} modules. Notation O_{Fn} means local quotient ring and function field.

(iii) Treat each $f_j : A_j \to A_{j+1}$ is homomorphism by natural morphism $\{f_{n-1}, f_n, f_{n+1}\} \to \{f_n, f_{n+1}, f_{n+2}\}$. If this condition is broken, we must modify F_j.

(iv) Define (X, A)={Collection of all (X_j, A_j)}.We introduce Zariski topology to (X, A) by open covering U_j and D_j that are defined as $U_j=\{p \mid f_j \notin p, p \in X\}$ and $D_j=\{p \mid F_j \notin p, p \in X\}$. Here $X=SpecA$={Collection of all $SpecA_j$}, $X_j=SpecA_j$. We find X_j is Noetherian locally because A_j is Noetherian.

Using above definitions, we can introduce sheaves of AFDCEs. It is known that sheaves by Ideals become coherent sheaves. Proper scheme over C, which is coherent sheaf, corresponds to some analytical scheme by GAGA. Especially projective scheme over C is proper scheme. Therefore validity of singularity confinement criterion is found. It is natural condition rather than criterion in projective space. Using this principle more actually, we show what implementation of AFDCE satisfies property of sheaf, especially coherent sheaf, and becomes proper scheme by following some examples.

Since $n \in Z$, collection of all A_n and $C_n=C[f_{n-1}, f_n, f_{n+1}], C[f_n, f_{n+1}, f_{n+2}], \ldots, C[f_{k-1}, f_k, f_{k+1}]$ are polynomials consist of infinite number of variables. Therefore Hilbert's basis theorem is not satisfied globally. That is, $SpecA$ has infinite elements and not Noetherian. Please remember that previous implementation (i) to (iv) satisfy coherent sheaf condition only locally (at every n). Therefore we need more conditions to construct entire coherent sheaf of AFDCE by this formulation.

For the condition of finite number of variables in entire space, we must add more condition for AFDCE. For example, forcing following condition gives Noetherian property of entire space of AFDCE,

(2) $$\begin{vmatrix} \frac{\partial F_n}{\partial f_{n-1}} & \frac{\partial F_n}{\partial f_{n+2}} \\ \frac{\partial F_{n+1}}{\partial f_{n-1}} & \frac{\partial F_{n+1}}{\partial f_{n+2}} \end{vmatrix} = \left| \frac{\partial F_n}{\partial f_{n-1}} \frac{\partial F_{n+1}}{\partial f_{n+2}} \right| \neq 0, \text{ for all } n,$$

or equivalently

$$\left| \frac{\partial F_n}{\partial f_{n-1}} \frac{\partial F_n}{\partial f_{n+1}} \right| \neq 0, \text{ for all } n.$$

Since by implicit function theorem, we can find following local relations at every n,

(3) $f_{n+1}=g_{n(+)}(f_n, f_{n-1}),\ f_{n-1}=g_{n(-)}(f_n, f_{n+1})$,

here $g_{n(+)}, g_{-n(-)}$ should be algebraic function and never spoil algebraic property of each F_n. We know that this condition can be modified to the case of $g_{n(+)}$ and $g_{-n(-)}$ are holomorphic, but we leave it. Then we can delete f_{n-1} and f_{n+2} from $C[f_{n-1}, f_n, f_{n+1}]$, $C[f_n, f_{n+1}, f_{n+2}]$ as $C[g_{n(-)}(f_n, f_{n+1}), f_n, f_{n+1}]$, $C[f_n, f_{n+1}, g_{n+1(+)}(f_n, f_{n+1})]$. Appling this condition for all F_n, we find all $C[f_{n-1}, f_n, f_{n+1}], C[f_n, f_{n+1}, f_{n+2}], \ldots, C[f_{k-1}, f_k, f_{k+1}], \ldots$ are included in the two variable polynomial $C[f_k, f_{k+1}]$ or holomorphic function. Since k is arbitrary, we can say $C[f_k, f_{k+1}]$ is germ at $(k, k+1)$ and also representation of solution function of AFDCE by germ at $(k, k+1)$.

It is easy to generalize this treatment for multi-step, several variable and simultaneous AFDCE. With this condition in this example, ($X=SpecA$, O_X(collection of A_n)) becomes coherent sheaf entirely. Expression of $C[f_j, f_{j+1}]$ *by* $C[f_k, f_{k+1}]$, $j \neq k$ is analogous to Taylor series representation $C[f_j, f_{j+1}]$ *by* $\{f_k, f_{k+1}\}$. In this case it corresponds functional series representation for near neighbor functions. We also found the condition (2) corresponds to preserving dimension of variables in each A_n. f_{n+2} in A_{n+1} takes over independency of f_{n-1} in A_n or initial conditions are preserved from A_n to A_{n+1}.

Definition: We call AFDCE that satisfies coherent sheaf conditions as coherent AFDCE (CAFDCE). We call these conditions as coherent condition for abbreviation. For general multi-step, several variables or simultaneous ADFCE, we define coherent condition as, (i) F_n gives coordinate ring, and F_n generates coverings of AFDCE as a non-singular algebraic manifold. Moreover A_n becomes Noetherian at every n. (ii) Existence of proper morphism $A_n \leftrightarrow A_{n+1}$ at every n, and every A_n satisfies coherent condition by Zariski topology [8]. (iii) Following dimensional condition is satisfied independently of n in each covering with regular coordinate system. $\dim(A_n)$=dim(Initial conditions or Boundary conditions) = Const.

Definition: We call singular point (set) of AFDCE where coherent condition is broken.

It is clear from the definition that CAFDCE has no singular point (set). In other words it becomes non-singular algebraic manifold using proper local coordinates.

2.1 IAFDCE example, Linear AFDCE

Consider n-sep linear AFDCE and its solution,

(4) $Cs_0 y + Cs_1 y_1 + Cs_2 y_2 + \cdots + Cs_n y_n = 0,$

(5) $c_0 y + c_1 y^1 + c_2 y^2 + \cdots + c_n y^n = 0\,,$

here Cs_j, c_j and y^k are Casoratian, integral constant and fundamental solutions. Usually integral constants are noted as C_j=-c_j/c_0 and Casoratian Cs_j is defined as $n \times n$ matrix determinant by eliminating column from following $(n+1) \times n$ matrix,

(6) $$\begin{bmatrix} y^1 & y^2 & y^3 & \cdots & y^n \\ y_1^1 & y_1^2 & y_1^3 & \cdots & y_1^n \\ y_2^1 & y_2^2 & y_2^3 & \cdots & y_2^n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ y_n^1 & y_n^2 & y_n^3 & \cdots & y_n^n \end{bmatrix}$$

Let F_n= $Cs_0 y + Cs_1 y_1 + Cs_2 y_2 + \cdots + Cs_n y_n$. It is clear A_n is Noetherian, because the number of elements y_j (j=1,..., n) are finite. Moreover F_n satisfies condition for finite number of total variables, because $\partial F_n / \partial y = Cs_0 \neq 0$ and $\partial F_n / \partial y_n = Cs_n \neq 0$. Conditions $Cs_0 \neq 0$ and $Cs_n \neq 0$ due to fundamental solutions y^j (j=1,..., n). By this property we can rewrite (4) as,

$y = -(Cs_1 y_1 + Cs_2 y_2 + \cdots + Cs_n y_n) / Cs_0$ or

$y_n = -(Cs_0 y_0 + Cs_1 y_1 + \cdots + Cs_{n-1} y_{n-1}) / Cs_n$. We found similarity between (4) and (5), comes directly from algebra of coherent property.

Algebraic singularity of F_n is given by $\partial F_n / \partial y = 0$ and $\partial F_n / \partial y_j = 0$, j=1,…,n. Clearly F_n is non-singular because $Cs_0 \neq 0$ and $Cs_n \neq 0$.

Linear AFDCE is a typical model equation that reflects coherent sheaf structure well. Note that Linear AFDCE satisfies finite algebraic property. We can easily confirm this property from,

(7) $$y_r = -\{h_0(Cs_0,...,Cs_{n-1}) y_s + h_1(Cs_0,...,Cs_{n-1}) y_{s+1} + \cdots + h_{n-1}(Cs_0,...,Cs_{n-1}) y_{s+n-1}\} / h_n(Cs_0,...,Cs_{n-1}).$$

In this case arbitral y_r can be represented by a general algebraic relation (7) with arbitral germ at {s, $s+1$,…, $s+n-1$}, and the number of germs, n, is independent of position.

3 Projective scheme in AFDCE

We introduced coherent condition into AFDCE. The sheaf condition include concept of connection between each covering.

Traditional ways of analysis for AFDCE also pay attention for the connection; although it hasignored the algebraic finite property until now. We never overlook investigation of finite property for AFDCE after this, because algebraic finite property is the main concept of coherent sheaf. In addition if we use GAGA to give integrability to AFDCE, we never overlook proper morphism property of AFDCE. It is known that morphism in projective space is proper morphism, therefore we don't need to pay attention to this property when we treat AFDCE in projective space. From this fact we found projective space is simple and convenient space for applying GAGA to AFDCE. In this section we review projective scheme shortly for this purpose. We can find more details from many textbooks of algebraic geometry.

We assume all AFDCEs in this section are homogeneous equations. As an example, using the same notation in previous section, we treat F_n in C_n. In this case $C_n = C[f_{n-1}; f_n; f_{n+1}; f_{o,n}]$ corresponds to polynomial function with complex coefficient in projective space. Then F_n is defined in the subspace of extended projective space by following treatment,

(8) $$\{f_{n-1}, f_n, f_{n+1}\} \xrightarrow[bijection]{} \{\frac{f_{n-1}}{f_{0,n}}; \frac{f_n}{f_{0,n}}; \frac{f_{n+1}}{f_{0,n}}; 1\},$$

here $f_{0,n} \neq 0$, $\{0;0;0;0\} \notin \{f_{n-1}; f_n; f_{n+1}; f_{0,n}\}$ and

(9) $$F_n \to (f_{0,n})^m F(\frac{f_{n-1}}{f_{0,n}}, \frac{f_n}{f_{0,n}}, \frac{f_{n+1}}{f_{0,n}}),\ 0 \le m,$$

when total order of $F(f_{n-1}, f_n, f_{n+1})$ equals m. By this treatment we can regard F_n in projective space as,

(10) $$B_n = (f_{0,n})^m F(\frac{f_{n-1}}{f_{0,n}}, \frac{f_n}{f_{0,n}}, \frac{f_{n+1}}{f_{0,n}}) = G_n(f_{n-1}; f_n; f_{n+1}; f_{0,n}),$$

here $B_n \in C[f_{n-1}; f_n; f_{n+1}; f_{o,n}]$ is homogeneous equation, and F_n= $G_n(f_{n-1}; f_n; f_{n+1}; 1)$. Therefore we can regard B_n as homogeneous ideal. For simplicity we assume B_n is a homogeneous prime ideal. We consider a space $Proj(PA_n)$ which consists of all homogeneous prime ideals except for irrelevant ideal in quotient ring PA_n= $S_n^{-1} C_n$, here S_n= $C[f_{n-1}; f_n; f_{n+1}; f_{o,n}] \setminus B_n$. We call this space PX_n=$Proj(PA_n)$. As the same manner in affine space, we can introduce Zariski topology locally using following definitions for open covering,

(11) D_j={p | $PA_j \notin p$, $p \in PX$}.

We also use affine covering U_j to cover D_i. In this case set of U_j is finer covering than set of D_i.

In $U_{0,i} \cap U_{0,j} \neq \Phi$ we treat,

(12) $(f_{0,i})^m F(\frac{f_{i-1}}{f_{0,i}}, \frac{f_i}{f_{0,i}}, \frac{f_{i+1}}{f_{0,i}})$

$= \frac{(f_{0,j})^m}{(f_{0,i})^m} (f_{0,i})^m F(\frac{f_{i-1}}{f_{0,i}}(\frac{f_{0,i}}{f_{0,j}}), \frac{f_i}{f_{0,i}}(\frac{f_{0,i}}{f_{0,j}}), \frac{f_{i+1}}{f_{0,i}}(\frac{f_{0,i}}{f_{0,j}}))$

$=(f_{0,j})^m F(\frac{f_{i-1}}{f_{0,j}}, \frac{f_i}{f_{0,j}}, \frac{f_{i+1}}{f_{0,j}})$.

We can treat inclusion F_n to projective space by different way from previous example, as following. In $U_{0,n-1} \cap U_{0,n} \cap U_{0,n+1} \neq \Phi$,

(13) $\{f_{n-1}, f_n, f_{n+1}\} \xrightarrow{bijection} \{\frac{f_{n-1}}{f_{0,n-1}}; \frac{f_n}{f_{0,n}}; \frac{f_{n+1}}{f_{0,n+1}};1;1;1\}$,

here $f_{0.n-1} \neq 0$, $f_{0.n} \neq 0$, $f_{0.n+1} \neq 0$,

$\{0;0;0;0;0;0\} \notin \{f_{n-1}; f_n; f_{n+1}; f_{0,n-1}; f_{0,n}; f_{0,n+1}\}$ and

(14) $F_n \rightarrow$

$(f_{0,n-1})^{m1}(f_{0,n})^{m2}(f_{0,n+1})^{m3} F(\frac{f_{n-1}}{f_{0,n-1}}, \frac{f_n}{f_{0,n}}, \frac{f_{n+1}}{f_{0,n+1}})$

$= G_n(f_{n-1}; f_n; f_{n+1}; f_{0,n-1}; f_{0,n}; f_{0,n+1})$,

$0 \le m1, m2, m3$, order of f_{n-1}, f_n, f_{n+1} in F.

Connection between each covering at $(U_{0,n-1} \cap U_{0,n} \cap U_{0,n+1}) \cap (U_{0,n} \cap U_{0,n+1} \cap U_{0,n+2}) \neq \Phi$ can be defined by the same way as previous example. We use notation PX={Collection of all $Proj(PA_j)$}. Note that $PX_n=Proj(PA_n)$ becomes finitely generated O_{Bn}-module, because B_n is defined by F_n and A_n, and A_n is clearly finitely generated O_{Fn}-module. A_n becomes finite covering of B_n. Moreover graded ring PA_n is Noetherian. Then $Proj(PA_n)$ becomes coherent sheaf at n locally.

We must add more condition to PX which becomes coherent sheaf globally in addition to (2). At the first we must define a rule how to choose proper $f_{0,j}$ for all j. Clearly we have no rule yet for selecting $f_{0,j}$ for all j. We must choose the total number of $f_{0,j}$ is finite. Instead $Proj(PA_n)$ becomes not finitely generated space. It maybe also proper choice for $f_{0,j}$ to make PA_n non-singular algebraic manifold, for example $f_{0,j}$ is defined from blowing-up at each j. We must also assume $f_{0,j}$ is finitely generated. For the purpose we assume another relation for example,

(15) $F_{o,n}(f_{0,n-1}, f_{0,n}, f_{0,n+1})=0$.

In addition, (10) or (14) also satisfies condition same to (2) and (3). More complex case is also considerable, for example

(16) $F_{o,n}(f_{0,n-1}, f_{0,n}, f_{0,n+1}, f_{n-1}, f_n, f_{n+1})=0$.

Shortly, (1) and (15) or (16) should form local connection for each f_j and $f_{0,j}$.

When these conditions are satisfied by new AFDCE system and each PX consists of finite number of generator by homogeneous element, then the new AFDCE system gives condition for $Proj(PA)$ which becomes coherent sheaf. In this case each PA_j becomes non-singular algebraic manifold with finite number of affine covering A_j, therefore we find PX by $\{f_{n-1}; f_n; f_{n+1}; f_{o,n}\}$ or $\{f_{n-1}; f_n; f_{n+1}; f_{0,n-1}; f_{0,n}; f_{0,n+1}\}$ becomes regular local ring.

Regular ring gives appropriate local parameters for the algebraic manifold; at last they span regular coordinate ring. In this example, dimension of each local base space at (n, $n+1$) is eight by { $f_{0,n-1}$, $f_{0,n}$, $f_{0,n+1}$, $f_{0,n+2}$, f_{n-1}, f_n , f_{n+1}, f_{n+2}} with four relations {G_n, G_{n+1}, $F_{0,n}$, $F_{0,n+1}$}=0. We expect $dim(Proj(PA_n))$ = $dim(Proj(PA_{n+1}))$ = 4 because { $f_{0,n}$, $f_{0,n+1}$, f_n , f_{n+1}} should become finite number of base element for germ at (n, $n+1$). It is clear $dim(ProjPA_n)=4$ also corresponds to number of integral constants or initial conditions at (n, $n+1$). In other words, arbitrary f_j and $f_{0,j}$ can be regarded as function in $C[f_n, f_{n+1}, f_{0,n}, f_{0,n+1}]$ $\in PA_n$ or holomorphic function by { f_n, f_{n+1}, $f_{0,n}$, $f_{0,n+1}$}.

We find divergence of some variables $\{f_{n-1}, f_n, f_{n+1}\}$ in AFDCE which can be properly treated by space $\{f_{n-1}; f_n; f_{n+1}; f_{o,n}\}$ by using local affine covering, because it is proper morphism by coherent condition. The divergence of AFDCE is found in only part of affine covering space. Note that resolution or blowing-up procedure is necessary to make above covering. At present we have no automatic blowing-up and down algorithm. Therefore algebraic entropy criterion becomes a kind of prescription for this problem at present.

4 Convergence and integrability of ANA

We treat orthodox numerical algorithm as a sample application using previous results. Durand -Kerner -Aberth method is numerical root finding algorithm for algebraic equation. Consider n-th degree algebraic equation with real number coefficient,

(17) $P(z) = z^n + a_1 z^{n-1} + \cdots + a_n = 0, a_n \neq 0$,

here $z \in C$. The n number of roots can be obtained numerically by following Newton's method,

$$z^{(k)} = \begin{bmatrix} z_1^{(k)} \\ \vdots \\ z_n^{(k)} \end{bmatrix}, f(z) = \begin{bmatrix} f_1(z_1,\cdots,z_n) \\ \vdots \\ f_n(z_1,\cdots,z_n) \end{bmatrix} = \begin{bmatrix} f_1(z) \\ \vdots \\ f_n(z) \end{bmatrix},$$

$$J(z) = (\partial f_i(z)/\partial z_j),$$

(18) $$z^{(k+1)} = z^{(k)} - J(z^{(k)})^{-1} f(z^{(k)}),$$

k is iteration number. Then (18) can be written as

(19) $$z_i^{(k+1)} = z_i^{(k)} - P(z_i^{(k)}) / \prod_{\substack{j=1 \\ j \neq i}}^{n} (z_i^{(k)} - z_j^{(k)}), i = 1,2,\ldots,n.$$

We can easily find that (19) is holomorphic mapping except for the case $z_i^{(k)} - z_j^{(k)} = 0$. Usually we can assume $z_i^{(k)} - z_j^{(k)} \neq 0$ at every iteration step, therefore we can regard (19) as holomorphic mapping at anytime. Clearly (17) has n numbers of constants which are equals to given a_j, *j=1,…, n*.
Especially a_1 is invariant for k, that is

(20) $$a_1 = -\sum_{j=1}^{n} z_j^{(k)} = -\sum_{j=1}^{n} z_j^{(r)}, k \neq r. \cdot$$

From the same algebraic treatment to AFDCEs, (19) gives n numbers of generators for ideals. It is clear that each equation for $z_j^{(k)} \to z_j^{(k+1)}$ (*j=1,…, n*) in (19) is independent, therefore they become generator of ideals. We can introduce open covering by Zariski topology as shown in Fig 1.

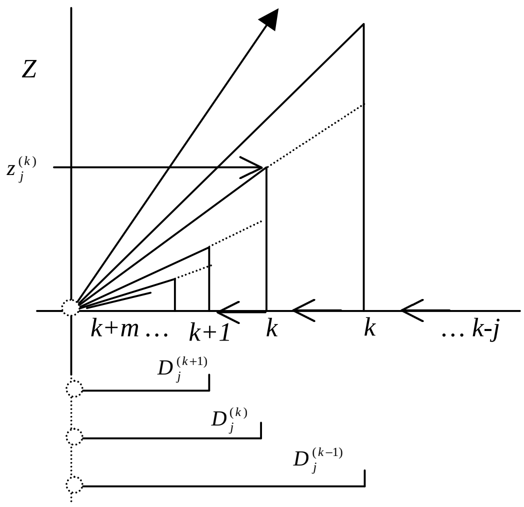

Fig 1. Algebraic view of Durand-Kerner-Aberth method

We can define restriction mapping on $D_j^{(k+1)} \subseteq D_j^{(k)}, D_j^{(k+1)} \cap D_j^{(k)} \neq f$ as,

(21) $$r_{j\;(k+1)}^{(k)} : D_j^{(k)} \to D_j^{(k+1)}$$

In this case, it is clear that mapping

(22) $$j_j^{(k)} : z_j^{(k)} \to z_j^{(k+1)}$$

is proper mapping [8, 12] whenever if $z_i^{(k)} - z_j^{(k)} \neq 0$ is satisfied. From these facts we can say that Durand-Kerner-Aberth method satisfies coherent and proper conditions. It is CAFDCE algorithm and generates integrable system step-by-step by self-integrable deformation. Moreover giving appropriate initial condition which grantees convergence corresponds to giving some deformed integrable system. This property may give superior convergent property of Durand-Kerner-Aberth method. Note that this deformation is not reversible as to k, because the deformation is contractive by convergence property.

5 Conclusion

Algebraic treatments of AFDCEs and ANAs are shown. It became clear that singularity confinement and algebraic entropy criteria are some parts of conditions of coherent and proper morphism conditions related to GAGA. Moreover sample AFDCEs which satisfy coherent condition and give proper morphism are given. By these samples, simple but actual treatment of AFDCEs and applying possibilities to analyze orthodox ANAs using proposed condition are shown.

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Appendix GAGA principle

Theorem 1 (*Serre*)*:* Let ***X*** be a proper (projective) scheme over ***C***. Then the functor *h* induces an equivalence of categories from the category of coherent sheaves on ***X*** to the category of coherent analytic sheaves on $\boldsymbol{X_h}$. Furthermore, for every coherent sheaf $\Im$ on ***X***, the natural maps

$$a_i : H^i(X, \Im) \longrightarrow H^i(X_h, \Im_h)$$

are isomorphisms, for all *i*.