Multilattices via Multisemilattices

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Abstract: M. Benado [1] and later D.J. Hansen [8] have offered an algebraic characterization of a multilattice, $(A, \leq)$, i.e., a poset, where every finite subset satisfies that any upper bound is greater than a minimal element of the upper bounds, and the dual property. To obtain this, they introduce some algebraic operators – denoted in this paper as $F_\wedge$ and $F_\vee$ –that are a generalization of the operators $\wedge$ and $\vee$ in a lattice However, both algebraic definitions present various serious disadvantages, derived from imposing the property of absorption onto $F_\wedge$ and $F_\vee$ and, more importantly, because these authors chose to substitute the associative property of $\wedge$ and $\vee$ in a lattice by another – which they consider to be a generalization of the associative property – which we will call *B-associativity* and *H-associativity*, respectively:

- These properties are, in both cases, a non-natural generalization of the associative property and, even more importantly, their definition cannot be fully disassociated from the properties specific to the order relation $\leq$.
- These properties do not take into account the behaviour of $F_\wedge$ and $F_\vee$ separately and, therefore, they do not allow us to offer an algebraic definition of *multisemilattices*.
- Furthermore, they do not allow the use of operators with flexible arity.

The usefulness of the multilattice structure considered by M. Benado in [1] is framed within the field of arithmetic. However, our interest in this structure is rather different and is driven by the search for efficient automated provers for temporal logic [3, 4].
In this work, we solve these disadvantages. Concretely, we introduce the structure of multisemilattices in algebraic terms, defining the *weak associativity property* for non-deterministic operators (i.e., operators of a set A in 2^A) of flexible arity. We prove that this property is independent from the B/H-associativity. The notion of algebraic multisemilattices we introduce here allows us to define the multilattices in algebraic terms and in a natural way. Specifically, we do this by taking as a starting point two algebraic multisemilattices $(A, F_\wedge)$ and $(A, F_\vee)$, adding the property of absorption. On the other hand, the algebraic definition provided makes these structures a natural generalization of the algebraic semilattices and lattices. In particular, we demonstrate that by substituting in our definition the property of *weak associativity* by the *associativity* for operators $F_\wedge$ and $F_\vee$, the multisemilattices and multilattices are reduced to semilattices and lattices, respectively.

Keywords: Poset; Non-deterministic operators; Multisemilattice; Multilattice; Universal Algebra.

1 Introduction

The results of our research group in the field of automated deduction [7, 6] are based on the efficient manipulation of unitary implicate and implicant sets. The main obstacle we face when attempting to extend the results obtained for classical logics [5] and many-valued logics [10] to temporal logics, is the greater complexity of the set of unitary formulae, or literals, with the *"logical implication"* relation, $(\texttt{Lit}, \trianglelefteq)$.

The first temporal logic we studied was the temporal propositional logic with linear and discrete Fnext time, where the unique fragment taken into account is the future fragment. In this kind of logic, the greater complexity of the set of literals was easily overcome because $(\texttt{Lit}, \trianglelefteq)$ has a lattice structure (Fig. 1).

However, when we add the past fragment in Fnext±, we find that $(\texttt{Lit}, \trianglelefteq)$ does not have a lattice structure, but it does preserve a large part of the lattice properties (Fig. 2.). The same applies to fully expressive temporal logics, such as US logic [9] and LN logic [2]. This made clear the need to carry out a theoretical study of new order structures that would enable us to work with the set of literals to obtain efficient methods of au-

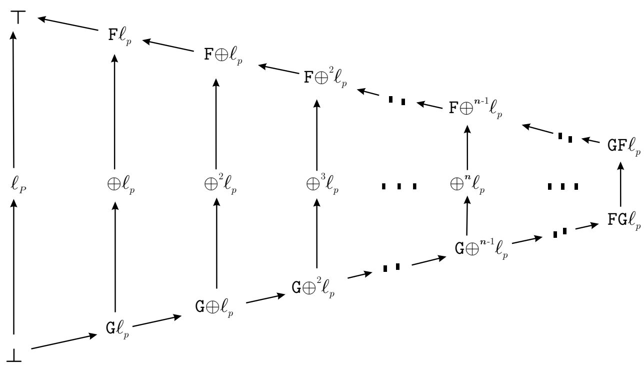


Figure 1: $(\texttt{Lit}, \trianglelefteq)$ in FNext

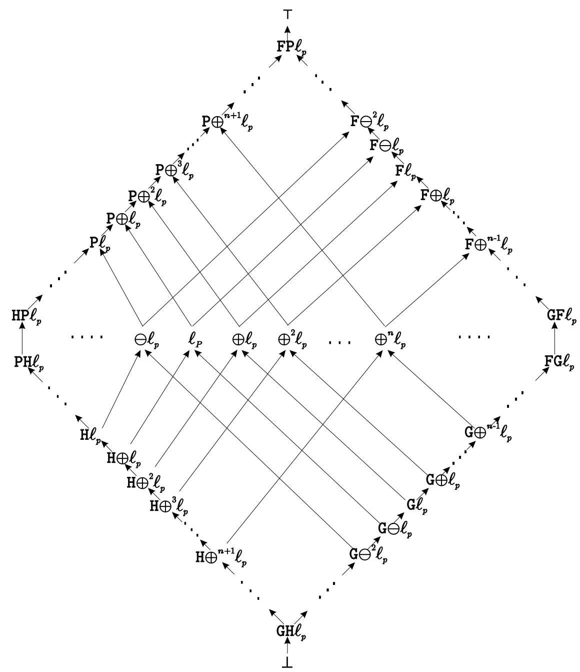


Figure 2: $(\texttt{Lit}, \trianglelefteq)$ in FNext±

tomated deduction for temporal logics. In these new structures, we substitute the concept of supremum and infimum with *multi-supremum* and *multi-infimum* that are the minimal elements of the upper bounds and the maximal elements of the lower bounds, respectively.

In these sets of literals the two following conditions are ensured:

(i) For every finite subset, any upper bound is greater or equal than a multi-supremum

(ii) and every lower bound is less or equal than a multi-infimum.

The partially ordered sets that fulfill this property receive the name *multilattices* and were introduced by Benado in [1]. This is then a generalization of lattices. However, for our purposes, we require,

- a generalization of semilattices that we call *multi-semilattices* and
- a good algebraic definition of both structures that would also allow us to use operators with flexible arity.

In the literature, we have found two algebraic definitions of multilattice, i.e., those proposed by M. Benado [1] and by D.J. Hansen [8]. To obtain them, they introduced some algebraic operators – denoted in this paper as $F_\wedge$ and $F_\vee$ –that are a generalization of the operators $\wedge$ and $\vee$ in a lattice. However, both algebraic definitions present some serious disadvantages, derived from imposing the property of absorption onto $F_\wedge$ and $F_\vee$ and, more importantly, because these authors chose to substitute the associative property of $\wedge$ and $\vee$ in a lattice by another – which they consider to be a generalization of the associative property – which we will call *B-associativity* and *H-associativity*, respectively:

- This property is, in both cases, a non-natural generalization of the associativity property and, even more importantly, its definition cannot be fully disassociated from the properties specific to the order relation $\leq$.
- These properties do not take into account the behaviour of $F_\wedge$ and $F_\vee$ separately and, therefore, they do not allow us to provide an algebraic definition of a *multisemilattice*.
- Furthermore, it does not allow the use of operators as flexible arity operators.

In this work, we solve these disadvantages by the algebraic characterization of properties (i) and, (ii). In other words, we define the structure of multisemilattices in algebraic terms. The starting point is to define a property – the *weak associativity* property – for non-deterministic operators (i.e., operators of a set A in 2^A) of flexible arity. The notion of algebraic multisemilattice we introduce here allow us to define multilattices in algebraic terms and in a natural way. Specifically, we do so by taking as a starting point two algebraic multisemilattices $(A, F_\wedge)$ and $(A, F_\vee)$ and adding the property of absorption. On the other hand, the algebraic definitions provided make these structures a natural generalization of algebraic semilattices and lattices. In particular, we prove that by substituting in our definition the property of *weak associativity* by the *associativity* for operators $F_\wedge$ and $F_\vee$, the multisemilattices and multilattices are reduced to semilattices and lattices, respectively.

This paper is structured as follows: Section 2 introduces the non-deterministic operators of flexible arity and their basic properties.

Section 3 describes the *weak associativity* property, and demonstrates that, contrary to what happens with B-associativity: every weakly associative deterministic operator is actually associative; the weak associativity and B-associativity properties are independent; and, finally, if F is a non-deterministic operator of flexible arity and weakly associative, and a has a symmetrical element, this symmetrical element is unique.

In Section 4 we define both the ordered and the algebraic structures of multisemilattices and we prove that both definitions are equivalent. Furthermore, we prove that a multisemilattice is a semilattice if and only if it is associative. In this section, we also study the subsets of a multisemilattice that are semilattices. Finally, we conclude by paying special attention to the notion of submultisemilattice and its characterization.

In Section 5 we define the ordered and the algebraic structures of multilattices and we relate them to multisemilattice structures.

In Section 6 we contrast our definition with those of Benado and Hansen (adapting the notations to the ones introduced in this work), to validate our statements in the introduction.

2 Non deterministic operators

In a partially ordered set, the set of multi-supremum and multi-infimum of a subset are not necessarily unitary. So, it is necessary to consider operators that have a set of elements of the domain as image.

Definition 2.1 *Let A be a set. We define the* **non-deterministic operator** *(ond[1]) of arity n in A, any total application $F: A^n \longrightarrow 2^A$.*
We define the **non-deterministic operator** *of flexible arity in A, any total application $F: A^* \longrightarrow 2^A$, where A^* is the universal language defined in A.*
If F is an ond with arity $\rho \in \mathbb{N} \cup \{*\}$ in A and $\varnothing \neq B \subseteq A$, we call **restriction of F to B**, *denoted by $F_{/B}$, to the ond in B given by $F_{/B}(\alpha) = F(\alpha) \cap B$. We say that F is* **full** *if $F(\alpha) \neq \varnothing$ for all $\alpha \in A^\rho$.*

Definition 2.2 *Let F be an ond of flexible arity in A. We say that*

1. *F is* **commutative** *if for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in A$ we have that $F(x_1, \ldots, x_n) = F(x_{\sigma 1}, \ldots, x_{\sigma n})$ for all permutations of n elements, σ.*

[1]In Spanish "operador no determinista".

2. *F is* **associative** *if for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in A$*

$$\begin{aligned} F(x_1, \ldots, x_n) &= F(F(x_1, \ldots, x_{n-1}), x_n) \\ &= F(x_1, F(x_2, \ldots, x_n)) \end{aligned}$$

3. *F is* **idempotent** *if $F(x, \overset{n}{\ldots}, x) = \{x\}$, for all $x \in A$ and all $n \in \mathbb{N}$.*

Definition 2.3 *Let F be an ond of flexible arity and $e \in A$. We say that e is a* **neutral element** *for F if for all $\omega \in A^*$ we have that $F(\omega) = F(\omega_1)$ where ω_1 it is the chain obtained when eliminating e from ω.*

Let F be a binary ond in A, $a \in A$, and $e \in A$ the neutral element. We say that $b \in A$ is the **symmetric element** *of a if $F(a, b) = F(b, a) = \{e\}$.*

As in the deterministic case, our interest lays in structures that have the absorption property. This will become patent in the algebraic characterization of the new ordered structures introduced in section 5.

Definition 2.4 *Let F and G be two onds of flexinle arity in A. We say that the pair (F, G) has the* **absorption** *property if for all $\omega \in A^*$ we have that:*

- *If $x \in \omega$, then $G(xy) = \{x\}$ for all $y \in F(\omega)$.*
- *If $x \in \omega$, then $F(xy) = \{x\}$ for all $y \in G(\omega)$.*

Notice that if (F, G) has the property of absorption, the two following conditions are satisfied:

(i') If $F(x_1, x_2) \neq \varnothing$, then $G(x_1, F(x_1, x_2)) = \{x_1\}$

(ii') If $G(x_1, x_2) \neq \varnothing$, then $F(x_1, G(x_1, x_2)) = \{x_1\}$

The two previous conditions, (i') and (ii'), are used to define the property of absorption by M. Benado in [1] for binary operators. However, as shown in the following example, these conditions are not enough:
Example 1.- We consider the binary onds F and G in the set $A = \{1, 2, 3, 4, 5\}$, given by the following tables:

F	1	2	3	4	5
1	$\{1\}$	$\{2,3,4,5\}$	$\{3\}$	$\{4\}$	$\{5\}$
2	$\{2,3,4,5\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{4\}$	$\{5\}$
4	$\{4\}$	$\{4\}$	$\{4\}$	$\{4\}$	$\{5\}$
5	$\{5\}$	$\{5\}$	$\{5\}$	$\{5\}$	$\{5\}$

G	1	2	3	4	5
1	$\{1\}$	$\varnothing$	$\{1\}$	$\{1\}$	$\{1\}$
2	$\varnothing$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$
3	$\{1\}$	$\{2\}$	$\{3\}$	$\{3\}$	$\{3\}$
4	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{4\}$
5	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$

(F, G) satisfies the conditions (i') and (ii'). However it doesn't satisfies (i) and (ii), because, although $G(1, F(1, 2)) = \{1\}$, we have that $2 \in F(1, 2)$ but $G(1, 2) = \varnothing$.

3 Choosing the associativity property

As we will see later, it is excessive to demand the associativity property to the o.n.d.s for the posets considered in this work. M. Benado in [1] introduces as associativity the following property for binary onds:

Definition 3.1 *Let F be a binary ond in A. F is* **B-associative** *if it satisfies:*

1. *Given $a, b, c \in A$ such that $F(a, b) \neq \varnothing$ and $F(F(a, b), c) \neq \varnothing$, then $F(b, c) \neq \varnothing$ and $F(a, F(b, c)) \neq \varnothing$.*
2. *For each $z \in F(F(a, b), c)$ exists $z' \in F(a, F(b, c))$ such that $F(z, z') = \{z\}$.*

On the other hand, D. J. Hansen in [8], introduces the following property for binary onds:

Definition 3.2 *Let F be a binary ond in A. F is* **H-associative** *if it satisfies:*
Given $a, b, c \in A$ such that $F(b, c) \neq \varnothing$, and $F(a, x) \neq \varnothing$ for some $x \in F(b, c)$. Then, $F(a, b) \neq \varnothing$ and $F(a, F(b, c)) \subseteq F(F(a, b), F(b, c))$

However, as we shall see, these properties are not a natural generalization of associativity, because if F is deterministic, they are not the associative property. For this reason, we introduce a new property which is weaker than the associative property.

Definition 3.3 *Let F be a binary ond in A. We say that F is* **weakly associative** *if for all $x_1, x_2, x_3, z \in A$ we have that: if $F(x_1, x_2) = \{z\}$, then:*

$$\begin{cases} F(F(x_1, x_2), x_3) \subseteq F(x_1, F(x_2, x_3)) \\ F(x_3, F(x_1, x_2)) \subseteq F(F(x_3, x_1), x_2) \end{cases}$$

Example 2.- The binary ond F in 2^U given by $F(A, B) = \{A \cup B, A \cap B\}$ is not associative, but is weakly associative.

Definition 3.4 *Let F be a binary ond in A. We say that F is* **weakly associative** *if for every $\alpha = \alpha_1\alpha_2\alpha_3 \in A^*$ with $\alpha_2 \neq \varepsilon$ and every $z \in A$ it satisfies that: if $F(\alpha_2) = \{z\}$, then*

$$F(\alpha_1 F(\alpha_2)\alpha_3) = \bigcap_{\substack{\alpha=\omega_1\omega_2\omega_3 \\ \omega_2\neq\varepsilon}} F(\omega_1 F(\omega_2)\omega_3)$$

Example 3.- Let $(A, \leq)$ be a poset. The ond of flexible arity $F(x_1, \ldots x_n) = Minimals\{x_1, \ldots x_n\}$ is not associative. However, it is weakly associative.

The following result is an immediate consequence from definition:

Lemma 3.5 *Let F be a deterministic operator. If F is weakly associative, then is associative.*

Theorem 3.6 *The B-associativity and the H-associativity properties are independent of the weak associativity.*

Proof 1 *Let us consider $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$ and the binary onds*

F	1	2	3
1	$\{2,3\}$	$\varnothing$	$\varnothing$
2	$\varnothing$	$\varnothing$	$\{1,2\}$
3	$\varnothing$	$\{1,2\}$	$\varnothing$

G	a	b	c
a	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{c\}$	$\{b\}$	$\{b\}$
c	$\{c\}$	$\{c\}$	$\{c\}$

F is weakly associative, but it is not B-associative, because: $F(1, 1) = \{2, 3\}$ and $F(F(1, 1), 3) = \{1, 2\}$, and however, $F(1, 3) = \varnothing$ and $F(1, F(1, 3)) = \varnothing$. Furthermore, F is not H-associative, because: $F(2, 3) = \{1, 2\}$ and $F(1, 1) = \{2, 3\}$ and however $F(1, 2) = \varnothing$.
G is not weakly associative, because: $\{c\} = G(G(b, b), a) \not\subseteq G(b, G(b, a)) = \{b\}$ However, G is B-associative, because:

- *$G(G(b, b), a) = \{c\}$, $G(b, G(b, a)) = \{b\}$ y $G(c, b) = \{c\}$*
- *$G(G(b, c), a) = \{c\}$, $G(b, G(c, a)) = \{b\}$ y $G(c, b) = \{c\}$*

Furthermore, G is also H-associative, because for all $x_1, x_2, x_3 \in B$, we always have that $F(x_1, F(x_2, x_3)) = F(F(x_1, x_2), F(x_1, x_3))$.

The previous example shows that, contrary to what happens with the weak associativity (see lemma 3.5), a B-associative deterministic operator is not necessarily associative. The following proposition ratifies us the kindness of the weak associative respect the B-associativity.

Proposition 3.7 *Let F be a binary ond in A, and e the neutral element of F. Then:*

(1) *If F is weakly associative, and $a \in A$ has symmetrical, this is unique.*

(2) *If F is B-associative, the unique element that has symmetrical is e.*

Now, we highlight a particular result of interest to the rest of the development:

Proposition 3.8 *Let F be an ond in A with flexible arity, weakly associative and idempotent. Then the three following conditions are satisfied:*

1. *For all $\omega \in A^*$, $F(\omega) = \bigcap_{\substack{\omega=\omega_1\omega_2\omega_3 \\ \omega_2\neq\varepsilon}} F(\omega_1 F(\omega_2)\omega_3)$*

2. For all $\omega = \alpha_1\alpha_2\alpha_3 \in A^*$, if $F(\alpha_2)$ is an unitary set, $F(\alpha_2) = \{z\}$, we have that $F(\omega) = F(\alpha_1 z \alpha_3)$

3. Given $\omega \in A^*$ and $z \in A$, if $F(x,z) = \{z\}$ for all $x \in \omega$ then $F(\omega z) = \{z\}$

Proof 2 *1. This is an immediate consequence of weak associativity and of the fact that, for the idempotent property, we have that $F(x) = \{x\}$ for all $x \in \omega$.*
2. Let $\omega = \alpha_1\alpha_2\alpha_3 \in A^*$ and $F(\alpha_2) = \{z\}$. Then,

$$F(\alpha_1 z\alpha_3) \stackrel{\dagger_1}{=} \bigcap_{\substack{\omega=\omega_1\omega_2\omega_3 \\ \omega_2\neq\varepsilon}} F(\omega_1 F(\omega_2)\omega_3) \stackrel{\dagger_2}{\subseteq} F(\omega) \stackrel{\dagger_3}{\subseteq} F(\alpha_1 z\alpha_3)$$

where in $\dagger_1$ we use weak associativity; in $\dagger_2$ idempotency and in $\dagger_3$ the item 1. Consequently, $F(\omega) = F(\alpha_1 z\alpha_3)$.
3. If the length of ω is 1, the result is obvious. Let us assume that the result is true for length n. If $\omega = \omega_1 x_1 \in A^*$ is a chain of length $n+1$ then, for all $x \in \omega$ we have that $F(x,z) = \{z\}$, in particular $F(x_1, z) = \{z\}$. Therefore,

$$F(\omega z) = F(\omega_1 x_1 z) \stackrel{\dagger_1}{=} F(\omega_1 F(x_1 z)) = F(\omega_1 z) \stackrel{\dagger_2}{=} \{z\}$$

where we use item 2 in $\dagger_1$, and the induction hypothesis in $\dagger_2$.

As a consequence of the previous lemma, we obtain the following result:

Corollary 3.9 *Let F be an ond with flexible arity, weakly associative, commutative and idempotent in a set A. Then, for all $\omega = x_1 \dots x_n \in A^*$ we have that $F(x_1 \dots x_n) = F(x_{n_1} \dots x_{n_k})$ where $x_{n_1} \dots x_{n_k} = \omega'$ is the chain obtained after eliminating the repetitions of elements in $\omega = x_1 \dots x_n$.*

4 Multisemilattices

We begin by introducing some concepts and results.

4.1 Multi-supremum and Multi-infimum

Definition 4.1 *Let $(A, \leq)$ be a poset. If $B \subseteq A$, we denote by $Cot^{\uparrow}(B)$ the set of upper bounds of B and by $Cot_{\downarrow}(B)$ the set of lower bounds of B. So, we have two operators $Cot^{\uparrow}, Cot_{\downarrow} : 2^A \longrightarrow 2^A$ defined as follows:*

$$Cot^{\uparrow}(B) = \bigcap_{b\in B} [b); \qquad Cot_{\downarrow}(B) = \bigcap_{b\in B} (b]$$

The next result is immediate.

Lemma 4.2 *Let $(A, \leq)$ be a poset and $(\,)\downarrow, (\,)\uparrow$ the down and up-closure operators. Then, operators $Cot^{\uparrow}$ and $Cot_{\downarrow}$ satisfy the following properties:*

1. $id_A \leq Cot_{\downarrow} \circ Cot^{\uparrow}$; $\quad id_A \leq Cot^{\uparrow} \circ Cot_{\downarrow}$
2. $Cot^{\uparrow} = Cot^{\uparrow} \circ (\,)\downarrow = (\,)\uparrow \circ\, Cot^{\uparrow}$;
   $Cot_{\downarrow} = Cot_{\downarrow} \circ (\,)\uparrow = (\,)\downarrow \circ\, Cot_{\downarrow}$.
3. $Cot^{\uparrow}(B) = \bigcup_{x\in Cot^{\uparrow}(B)} [x)$; $Cot_{\downarrow}(B) = \bigcup_{x\in Cot_{\downarrow}(B)} (x]$

Our interest focuses on ordered structures where unions of the kind $\bigcup_{x\in\Gamma}[x)$ and $\bigcup_{x\in\Gamma}(x]$ play a relevant role.

Proposition 4.3 *Let $(A, \leq)$ be a poset. Then $Minimals(X) = Minimals(X\uparrow)$ for all $X \subseteq A$.*

Definition 4.4 *Let $(A, \leq)$ be a poset, $a \in A$ and $B \subseteq A$.*

- *A* **multi-supremum of** *B is a minimal element of $Cot^{\uparrow}(B)$. We denote by $Multi\text{-}sup(B)$ the set of multi-supremum of B.*
- *A* **multi-infimum of** *B is a maximal element of $Cot_{\downarrow}(B)$. We denote by $Multi\text{-}inf(B)$ the set of multi-infimum of B.*

4.2 Ordered Multisemilattices

In this section we introduce the concept of multisemilattices ase ordered structures.

Definition 4.5 *An* **ordered $\vee$-multisemilattice** *is a poset, $(A, \leq)$, such that for every nonempty finite subset, $H \subseteq A$ we have that:*

$$Cot^{\uparrow}(H) = \bigcup\{[z) \mid z \in \text{ Multi-sup } (H)\}$$ [2]

For duality, we obtain the definition of **ordered $\wedge$-multisemilattice***.*

The following proposition allow us to provide an equivalent definition.

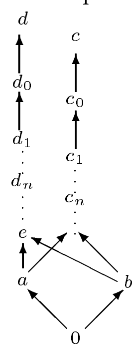
Proposition 4.6 *A poset, $(A, \leq)$, is an ordered $\vee$-multisemilattice if and only if for every nonempty finite subset H of A the following condition is satisfied:*

If $x \in Cot^{\uparrow}(H)$, then exists $z \in$Multi-sup(H) such that $z \leq x$

By duality, we obtain the result for $\wedge$-multisemilattices.

[2] Notice that we don't need that $Cot^{\uparrow}(H) \neq \varnothing$.

EXAMPLE 4.- Let A be the poset whose diagram is:



Diag$_1$

This poset is a $\wedge$-multisemilattice, but it is not a $\vee$-multisemilattice. Indeed, given an arbitrary c_i, we have that $c_i \in Cot^{\uparrow}(\{a,b\})$, but there is not any $z \in$ Multi-sup$(\{a,b\})$ that fulfils $z \leq c_i$.

4.3 Algebraic Multisemilattice

In order to introduce the algebraic characterization of multisemilattice, we define new specific properties for onds.

Definition 4.7 *Let F be an ond with flexible arity in A. We say that F has the property of* **comparability** *if for all $\omega \in A^*$ the two following conditions are satisfied:*
comp$_1$*: if $z \in F(\omega)$, then $F(x,z) = \{z\}$ for all $x \in \omega$.*
comp$_2$*: if $z_1, z_2 \in F(\omega)$ and $F(z_1,z_2) = \{z_1\}$, then $z_1 = z_2$.*

Proposition 4.8 *Let F be an ond in A of flexible arity, weakly associative, commutative and idempotent. If F satisfies the property of comparability, then*

$$z \in F(\omega) \text{ iff } F(\omega) = \{z\} \text{ for all } \omega \in A^* \text{ and } z \in \omega \quad (1)$$

Proof 3 *It is obvious that (1) is sufficienct. Let us prove the necessity. From corollary 3.9 we can assume that $x \neq y$ for all $x, y \in \omega$. We have that, if $z \in F(\omega)$:*

$$\{z\} \overset{\dagger_1}{=} F(\omega z) \overset{\dagger_2}{=} F(\omega)$$

where in $\dagger_1$ we have made use of **comp**$_1$ *and item 2 of lemma 3.8, and in $\dagger_2$ we have made use of corollary 3.9.*

The following example shows that the reciprocal is not true.

EXAMPLE 5.- Let us consider the binary ond given in the following table:

F	a	b	c	d
a	$\{a\}$	$\{c,d\}$	$\{b,d\}$	$\{b,c\}$
b	$\{c,d\}$	$\{b\}$	$\{a,d\}$	$\{a,c\}$
c	$\{b,d\}$	$\{a,d\}$	$\{c\}$	$\{b,a\}$
d	$\{b,c\}$	$\{c,a\}$	$\{b,a\}$	$\{d\}$

F is commutative, idempotent, weakly associative and it satisfies the property (1) given in the previous proposition, but not the comparability because, for example, $c \in F(a,b)$ but $\{c\} \neq F(a,c) = \{b,d\}$ and, consequently, F does not satisfies **comp**$_1$. Therefore, the ond of flexible arity F' defined from F as: $F'(a) = \{a\}$ and $F'(a\omega) = F'(aF'(\omega))$ for all $\omega \in \{a,b,c,d\}^*$ is the counter-example we were looking for.

Definition 4.9 *Let $(A, \leq)$ be an ordered $\vee$-multisemilattice. We define the ond of flexible arity, $F_\vee$, in A by $F_\vee(x_1,\ldots,x_n) =$ Multi-sup $(\{x_1,\ldots,x_n\})$.*

Dually, let $(A, \leq)$ be an ordered $\wedge$-multisemilattice and we define the ond $F_\wedge$ in A by $F_\wedge(x_1,\ldots,x_n) =$ Multi-inf $(\{x_1,\ldots,x_n\})$.

From the definitions of $F_\vee$ and $F_\wedge$, we obtain the following result:

Proposition 4.10 *Let $(A, \leq)$ be an ordered $\odot$-multisemilattice. Then, the ond $F_\odot$ satisfies the commutative, idempotent and comparability properties.*

The following theorem establishes the more outstanding properties of $F_\vee$ and $F_\wedge$.

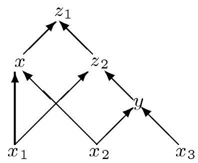
Theorem 4.11 *Let $(A, \leq)$ be an ordered $\odot$-multisemilattice. Then, given $\omega = \alpha_1\alpha_2 \in A^*$ where $\alpha_2 \neq \varepsilon$ we have that:*

1. $F_\vee(\omega) = Minimals F_\vee(\alpha_1 F_\vee(\alpha_2))$ *and* $F_\wedge(\omega) = Maximals F_\wedge(\alpha_1 F_\wedge(\alpha_2))$.
2. $F_\odot(\omega) \subseteq \bigcap_{\substack{\omega=\omega_1\omega_2 \\ \omega_2 \neq \varepsilon}} F_\odot(\omega_1 F_\odot(\omega_2))$

Proof 4 *To prove $F_\vee(\omega) = Minimals F_\vee(\alpha_1 F_\vee(\alpha_2))$, from proposition 4.3, it is enough to prove that $F_\vee(\alpha_1 F_\vee(\alpha_2)) \uparrow = Cot^{\uparrow}(\omega)$. The inclusion $F_\vee(\alpha_1 F_\vee(\alpha_2))\uparrow \subseteq Cot^{\uparrow}(\omega)$ is evident. If $z \in Cot^{\uparrow}(\omega)$ then, by definition of $F_\vee$, there exists $z_1 \in F_\vee(\alpha_2)$ such that $z_1 \leq z$ and $z \in Cot^{\uparrow}(\alpha_1 z_1)$. Again by the definition of $F_\vee$, $z_2 \in F_\vee(\alpha_1 z_1) \subseteq F_\vee(\alpha_1 F_\vee(\alpha_2))$ exists, where $z_2 \leq z$. Therefore $z \in F_\vee(\alpha_1 F_\vee(\alpha_2))\uparrow$. The proof for $F_\wedge$ is similar. The item 2 is immediate from 1.*

The example given below shows that, in general, the ond $F_\odot$ is not associative in an ordered $\odot$-multisemilattice .

EXAMPLE 6.- The following diagram corresponds to a $\vee$-multisemilattice where $F_\vee$ is not associative.



Diag$_2$

because $F_\vee(x_1, F_\vee(x_2, x_3)) = \{z_2\}$ and $F_\vee(F_\vee(x_1, x_2), x_3) = \{z_1, z_2\}$.

Proposition 4.12 *Let $(A, \leq)$ be an ordered $\odot$-multisemilattice. Then, $F_\odot$ is weakly associative.*

Proof 5 *We prove it for $\odot = \vee$. Given that $F_\vee$ is commutative, it is enough to prove that given $\omega = \alpha_1\alpha_2 \in A^*$ where $\alpha_2 \neq \varepsilon$, if $F_\vee(\alpha_2) = \{z\}$, then:*

$$F_\vee(\alpha_1 F_\vee(\alpha_2)) = \bigcap_{\substack{\omega=\omega_1\omega_2\\ \omega_2\neq\varepsilon}} F_\vee(\omega_1 F_\vee(\omega_2))$$

Since $F_\vee(\alpha_2) = \{z\}$, we have that $F_\vee(\alpha_1 F_\vee(\alpha_2)) = Minimals F_\vee(\alpha_1 F_\vee(\alpha_2))$ and, in consequence, the result is an immediate consequence of theorem 4.11.

Now we can provide the definition of an algebraic multisemilattice.

Definition 4.13 *An* **algebraic multisemilattice**, *(A, F), is a set A with an ond F of flexible arity in A, that satisfies the following properties:*

$(MSR1)$ *Commutative law.*
$(MSR2)$ *Weakly associative law.*
$(MSR3)$ *Idempotency law.*
$(MSR4)$ *Comparability law.*

Theorem 4.14

i) Let $\mathbb{M}_{s\vee} = (A, \leq)$ be an ordered $\vee$-multisemilattice. Then, $(A, F_\vee)$ where $F_\vee(x_1, \ldots, x_n) =$ Multi-sup $(\{x_1, \ldots, x_n\})$ is an algebraic $\vee$-multisemilattice, denoted by $\mathbb{M}^a_{s\vee}$.

ii) Let $\mathbb{M}_{s\vee} = (A, F_\vee)$ be an algebraic $\vee$-multisemilattice. The set A with the order relation "$x \leq y$ if and only if $F_\vee(x, y) = \{y\}$" is an ordered $\vee$-multisemilattice, denoted by $\mathbb{M}^o_{s\vee}$.

For duality, the result is obtained for $\wedge$-multisemilattices.

Proof 6 *i) Let us assume that $\mathbb{M}_{s\vee} = (A, \leq)$ is an ordered $\vee$-multisemilattice. The propositions 4.10 and 4.12 ensure that $F_\vee$ verifies the axioms in definition 4.13 and, therefore, $\mathbb{MS}^a_\vee = (A, F_\vee)$ is an algebraic $\vee$-multisemilattice.*

ii) Conversely, let us assume that $\mathbb{M}_{s\vee} = (A, F_\vee)$ is an algebraic $\vee$-multisemilattice. Firstlt, let us see that $\leq$ is an order relation:

- The idempotency law ensures that $\leq$ is reflexive.

- The commutative law ensures that $\leq$ is antisymmetric.

- $\leq$ is transitive because, for all $x, y, z \in A$, if $x \leq y \leq z$, then $F_\vee(x, y) = \{y\}$ and $F_\vee(y, z) = \{z\}$ and, from weak associativity, $F_\vee(F_\vee(x, y), z) = F_\vee(x, F_\vee(y, z))$, that is, $\{z\} = F_\vee(x, z)$.

Let us prove that, if $x \in Cot^\uparrow(\{x_1, x_2\})$, then there exists $z \in A$, such that $z \leq x$ and $z \in F_\vee(x_1, x_2)$. Based on the hypothesis, $F_\vee(x_1, x) = \{x\}$ and $F_\vee(x_2, x) = \{x\}$ and, item 2 of lemma 3.8, we have that $F_\vee(x_1, x_2, x) = \{x\}$. Therefore, according to the property of weak associativity, $\{x\} \subseteq F_\vee(F_\vee(x_1, x_2), x)$, that is, there exists $z \in F_\vee(x_1, x_2)$ such that $x \in F_\vee(z, x)$ and, from proposition 4.8, $F_\vee(z, x) = \{x\}$, so, $z \leq x$.

Theorem 4.15

i) If $\mathbb{M}_{s_\odot} = (A, \leq)$ is an ordered $\odot$-multisemilattice, then $(\mathbb{M}^a_{s_\odot})^o = \mathbb{M}_{s_\odot}$.

ii) If $\mathbb{M}_{s_\odot} = (A, F_\odot)$ is an algebraic $\odot$-multisemilattice, then $(\mathbb{M}^o_{s_\odot})^a = \mathbb{M}_{s_\odot}$.

Proof 7 *The item i) is immediate. Let $\mathbb{M}_{s\vee} = (A, F_\vee)$ be an algebraic $\vee$-multisemilattice and $(\mathbb{MS}^o_\vee)^a = (A, F'_\vee)$. Firstly, we will prove that $F_\vee \subseteq F'_\vee$. If $z \in F_\vee(\omega)$, by the comparability law, $F_\vee(x, z) = \{z\}$ for all $x \in \omega$ and so $z \in Cot^\uparrow_{\leq_{F_\vee}}(\omega)$. Next, we prove that z is a minimal element of the set $Cot^\uparrow_{\leq_{F_\vee}}(\omega)$. If $z_1 \in Cot^\uparrow_{\leq_{F_\vee}}(\omega)$ where $z_1 \leq z$, we can ensure that $z_2 \in F_\vee(\omega)$ exists, such that $z_2 \leq z_1 \leq z$. Then, by comparability, $z_2 = z$ and, consequently, $z \in F'_\vee(\omega)$.*

Finally, we prove that $F'_\vee \subseteq F_\vee$, that is, if z is a minimal element in $Cot^\uparrow_{\leq_{F_\vee}}(\omega)$, then $z \in F_\vee(\omega)$. Since z is an upper bound (respect to the order $\leq_{F_\vee}$) of ω, we have that $F_\vee(x, z) = \{z\}$ for all $x \in \omega$ and, therefore, there exists $z_1 \in F_\vee(\omega)$ where $z_1 \leq z$. On the other hand, the comparability law ensures that $z_1 \in Cot^\uparrow_{\leq_{F_\vee}}(\omega)$ and, as z is a minimal element of this set, $z = z_1$.

Example 7.- Let us consider the poset given in example 4 whose diagram is **Diag**$_1$. This poset is not an ordered $\vee$-multisemilattice. Indeed, the ond $F_\vee$ is not weakly associative: $F_\vee(a, F_\vee(b, c)) = \{c\} \not\subseteq F_\vee(F_\vee(a, b), c) = \varnothing$.

4.4 Associative Multisemilattice

In this section we prove that, as indicated in the introduction, the presence of associativity reduces multisemilattices to semilattices.

Definition 4.16 *Let $(A, F_\odot)$ be a $\odot$-multisemilattice. We say that is* **associative** *if $F_\odot$ has the associative property.*

Lemma 4.17 *Let $(A, F_\odot)$ be a $\odot$-multisemilattice. A is associative if and only if $|F_\odot(\omega)| \leq 1$ for all $\omega \in A^*$.*

An immediate consequence of the previous lemma is the following theorem.

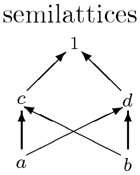
Theorem 4.18 *Let $(A, F_\odot)$ be a $\odot$-multisemilattice. Then A is a $\odot$-semilattice if and only if $F_\odot$ is associative and full.*

Moreover, if $(A, F_\odot)$ is a bounded $\odot$-multisemilattice, A is a $\odot$-semilattice if and only if $F_\odot$ has the associative property.

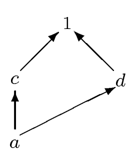
4.5 Submultisemilattices

Obviously, in a multisemilattice, we can found semilattices as shows the following example. This justify the name given to this structure.

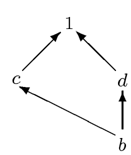
EXAMPLE 8.- In the following diagrams, $\mathbf{Diag}_3$ represents a $\vee$-multisemilattice A, and $\mathbf{Diag}_4$ and $\mathbf{Diag}_5$ are semilattices of A.



$\mathbf{Diag}_3$



$\mathbf{Diag}_4$



$\mathbf{Diag}_5$

As usual, we can give the following definition:

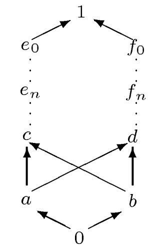
Definition 4.19 *Given a $\odot$-multisemilattice $(A, F_\odot)$ and a subset $\varnothing \neq B \subseteq A$, we say that B is $\odot$-***submultisemilattice** *of A if the restriction of $F_\odot$ to B, $F_{\odot/B}$, provides the structure of $\odot$-multisemilattice to B.*

A first approach to characterize submultisemilattices, could be the following:

A subset B of a $\odot$-multisemilattice $(A, F_\odot)$ is a $\odot$-submultisemilattice of A if for all $\omega \in B$ such that $F_\odot(\omega) \neq \varnothing$ we have that $F_\odot(\omega) \cap B \neq \varnothing$.

However this characterization is not the appropriate one, as we can see in the following example:

EXAMPLE 9.- Let $(A, \leq)$ be a multisemilattice whose diagram is:



$\mathbf{Diag}_6$

and $B = A - \{d\}$. Although B satisfies that $F_\odot(\{x, y\}) \cap B \neq \varnothing$ for all $x, y \in B$, it is not a $\odot$-multisemilattice.

Theorem 4.20 (Submultisemilattices characterization) *Let $(A, F_\odot)$ be a $\odot$-multisemilattice, $\varnothing \neq B \subseteq A$ and $F'_\odot$ the restriction to B of $F_\odot$. Then, B is a $\odot$-submultisemilattice if and only if for all $a, b, x \in B$,*

$$F'_\odot(a, x) = F'_\odot(b, x) = \{x\} \text{ implies that } x \in F'_\odot(F'_\odot(a, b), x)$$

Notice that the previous characterization means that if, when checking the table given by the operator $F'_\vee$, we observe that if $F'_\vee(b_1, x) = F'_\vee(b_2, x) = \{x\}$, then there is an element $y \in F'_\vee(b_1, b_2)$ such that $F'_\vee(y, x) = \{x\}$.

$F'_\vee$	$\cdots$	x	$\cdots$	b_2	$\cdots$
$\vdots$					
b_1	$\cdots$	x	$\cdots$	$\{y, \cdots\}$	$\cdots$
$\vdots$					
b_2	$\cdots$	x	$\cdots$	$\cdots$	$\cdots$
$\vdots$					
y	$\cdots$	x	$\cdots$	$\cdots$	$\cdots$
$\vdots$					

EXAMPLE 10.- Let us consider $A = \{a, b, c, d, e\}$, $B_1 = \{a, b, c\}$ and $B_2 = \{a, b, e\}$ and the following onds:

$F_\vee$	a	b	c	d	e
a	$\{a\}$	$\{c, d\}$	$\{c\}$	$\{d\}$	$\{e\}$
b	$\{c, d\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	$\{e\}$	$\{e\}$
d	$\{d\}$	$\{d\}$	$\{e\}$	$\{d\}$	$\{e\}$
e	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$

$F_{\vee/B_1}$	a	b	c
a	$\{a\}$	$\{c, d\}$	$\{c\}$
b	$\{c, d\}$	$\{b\}$	$\{c\}$
c	$\{c\}$	$\{c\}$	$\{c\}$

$F_{\vee/B_2}$	a	b	e
a	$\{a\}$	$\varnothing$	$\{e\}$
b	$\varnothing$	$\{b\}$	$\{e\}$
e	$\{e\}$	$\{e\}$	$\{e\}$

$(B_1, F_{\vee/B_1})$ is a submultisemilattice of $(A, F_\vee)$, while $(B_2, F_{\vee/B_2})$ is not a submultisemilattice of $(A, F_\vee)$.

5 Multilattices

We have now all the neccesary elements to discuss the study of ordered structure which is a generalization of the lattice structure.

5.1 Ordered Multilattices

Definition 5.1 *An* **ordered multilattice** *is a poset, $(A, \leq)$, such that for every nonempty finite $H \subseteq A$ we have that:*

$$Cot^{\uparrow}(H) = \bigcup\{[z) \mid z \in \text{Multi-sup}(H)\}$$

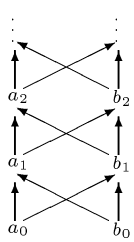
$$Cot_{\downarrow}(H) = \bigcup\{(z] \mid z \in \text{Multi-inf}(H)\}$$

The following lemma, of immediate demonstration, enable us to provide an equivalent definition of an ordered multilattice.

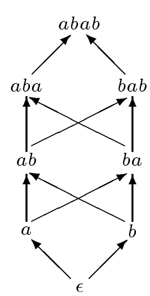
Lemma 5.2 *A poset, $(A, \leq)$, is an ordered multilattice if and only if for every nonempty finite subset H of A the following conditions are satisfied:*

- *If $x \in Cot^{\uparrow}(H)$, then exists $z \in$ Multi-sup(H) such that $z \leq x$.*
- *If $x \in Cot_{\downarrow}(H)$, then exits $z \in$ Multi-inf(H) such that $x \leq z$.* [3]

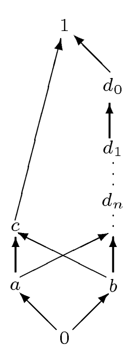
Example 11.- Diagram $\mathbf{Diag}_7$ is an infinite multilattice, and diagram $\mathbf{Diag}_8$ shows the set of subchains of "*abab*", which is a multilattice but not is a lattice.



$\mathbf{Diag}_7$



$\mathbf{Diag}_8$



$\mathbf{Diag}_9$

5.2 Algebraic Multilattices

Definition 5.3 *Let $(A, \leq)$ be an ordered multilattice. We define $F_{\vee}(x_1, \ldots x_n) =$ Multi-sup$\{x_1, \ldots x_n\}$ and $F_{\wedge}(x_1, \ldots x_n) =$ Multi-inf$\{x_1, \ldots x_n\}$ in A.*

[3] This is the definition of multilattice given by Hansen en [8]

The next lemma is an immediate consequence of the ordered multilattice definition:

Lemma 5.4 *Let $(A, \leq)$ be an ordered multilattice. Then, $(A, \leq)$ is a lattice if and only if for all $\omega \in A^* \smallsetminus \{\varepsilon\}$ we have that $F_{\vee}(\omega)$ and $F_{\wedge}(\omega)$ are unitary sets.*

$\mathbf{Diag}_9$ shows that in the previous lemma is not possible to make weaker the hypothesis that $(A, \leq)$ is an ordered multilattice, that is, if A is an arbitrary poset and $F_{\vee}$ and $F_{\wedge}$ have as images unitary sets, we can not ensure that A is a lattice neither a multilattice.

Lemma 5.5 *Let $(A, \leq)$ be an ordered multilattice. Then the pair $(F_{\vee}, F_{\wedge})$ satisfies the absorption law.*

Definition 5.6 *An* **algebraic multilattice***, $(A, F_{\vee}, F_{\wedge})$, is a set A with two onds $F_{\vee}$ and $F_{\wedge}$ in A, that verify the following axioms:*

$(MR1)$ *Commutative laws.*
$(MR2)$ *Weakly associative laws.*
$(MR3)$ *Idempotency laws.*
$(MR4)$ *Comparability laws.*
$(MR5)$ *Absorption law.*

Lemma 5.7 *Let F and G be two onds in A such that they verify the absorption law. Then:*

1. *$F(a, b) = \{a\}$ if and only if $G(a, b) = \{b\}$.*
2. *F and G satisfy the* $\mathbf{comp}_1$ *property.*

Proof 8 *1. is immediate. The proof of 2 is as follows: If $z \in F(\omega)$, then according to the absorption property we have that $G(x, z) = \{x\}$ for all $x \in \omega$, and according to item 1, if we apply again the absorption property, we have that $F(x, z) = \{z\}$ for all $x \in \omega$.*

Notice that from this result, we can modify the definition 5.6, by substituting the laws of comparability given in axiom $(MR4)$, by $\mathbf{comp}_2$.

Theorem 5.8

i) *Let $\mathbb{M} = (A, \leq)$ be an ordered multilattice. Then $(A, F_{\vee}, F_{\wedge})$ is an algebraic multilattice denoted by $\mathbb{M}^a$, being $F_{\vee}(x_1, \ldots x_n) =$ Multi-sup$\{x_1, \ldots x_n\}$ and $F_{\wedge}(x_1, \ldots x_n) =$ Multi-inf$\{x_1, \ldots x_n\}$.*

ii) *Let $\mathbb{M} = (A, F_{\vee}, F_{\wedge})$ be an algebraic multilattice. The set A with the order relation "$x \leq y$ if and only if $F_{\vee}(x, y) = \{y\}$" [4] is an ordered multilattice, denoted by $\mathbb{M}^o$.*

iii) *Given the ordered multilattice $\mathbb{M} = (A, \leq)$, we have that $(\mathbb{M}^a)^o = \mathbb{M}$.*

[4] Or, by lemma 5.7, $x \leq y$ if and only if $F_{\wedge}(x, y)) = \{x\}$

iv) Given the algebraic multilattice $\mathbb{M} = (A, F_\vee, F_\wedge)$, we have that $(\mathbb{M}^o)^a = \mathbb{M}$.

Proposition 5.9 *Let $(A, F_\vee)$ and $(A, F_\wedge)$ be multisemilattices. Then, $(A, F_\vee, F_\wedge)$ is a multilattice if and only if $(F_\vee, F_\wedge)$ it satisfies the property of absorption*

5.3 Associative multilattices

In this section we will prove that, as in the multisemilattices case, associativity reduces the multilattices to lattices.

Definition 5.10 *Let $(A, F_\vee, F_\wedge)$ be a multilattice. We say that A is an* **associative** *multilattice if $F_\vee$ and $F_\wedge$ satisfy the associative property.*

Theorem 5.11 *Let $(A, F_\vee, F_\wedge)$ be a full multilattice. Then, the following conditions are equivalent: 1) $F_\vee$ is associative; 2) $F_\wedge$ is associative; 3) $(A, F_\vee, F_\wedge)$ is a lattice.*

Proof 9 *It is immediate to prove that item 3) implies items 1) and 2). To prove that item 1) implies item 3) (and, by duality, that item 2) implies item 3)) it is sufficient to verify that if $F_\vee$ is associative and full, then $F_\vee(a, b)$ and $F_\wedge(a, b)$ are unitary sets for any $a, b \in A$. $F_\vee(a, b)$ is unitary by lemma 4.17. Let us see that $F_\wedge(a, b)$ is also unitary:*

Since $F_\wedge(a, b) \neq \varnothing$, if $c, d \in F_\wedge(a, b)$, we have that $c, d \in Maximal(Cot_\downarrow(\{a, b\}))$. On the other hand, $a, b \in Cot^\uparrow(\{c, d\})$ and, since $F_\vee(c, d)$ is unitary, $x = \sup(\{c, d\})$ exists, and therefore $x \in Cot_\downarrow(\{a, b\})$. Also, since $c \le x$ and $d \le x$, the maximality of c and d ensures that $c = x = d$. Finally, the lemma 5.4 ensures that A is a lattice.

Corollary 5.12 *Let $(A, F_\vee, F_\wedge)$ be a bounded multilattice. A is a lattice if and only if, is associative.*

5.4 Submultilattices

Definition 5.13 *Let $(A, F_\vee, F_\wedge)$ be a multilattice. We say that $\varnothing \neq B \subseteq A$ is a* **submultilattice** *of A if $(B, F_{\vee/B}, F_{\wedge/B})$ is a multilattice.*

Theorem 5.14 (Submultilattices characterization) *Let $(A, F_\vee, F_\wedge)$ be a multilattice, and $\varnothing \neq B \subseteq A$. Then, B is a submultilattice if and only if for all $a, b, x \in B$ the two following conditions are satisfied:*

1. *$F_{\vee/B}(a, x) = F_{\vee/B}(b, x) = \{x\}$ implies that $x \in F_{\vee/B}(F_{\vee/B}(a, b), x)$*
2. *$F_{\wedge/B}(a, x) = F_{\wedge/B}(b, x) = \{x\}$ implies that $x \in F_{\wedge/B}(F_{\wedge/B}(a, b), x)$*

6 Comparative Study

We finish this paper by comparing our definitions with those of Benado in [1] and Hansen in [8] (but adapting the notations to those introduced in this work), in order to confirm what we stated in the introduction.

Definition 6.1 (Benado[1]) *An* **algebraic multilattice**, *$(A, F_\vee, F_\wedge)$, is a set A with two operators $F_\vee : A \times A \to 2^A$ and $F_\wedge : A \times A \to 2^A$, that satisfies: ($\mathfrak{M}$I) Idempotency laws, ($\mathfrak{M}$II) B-associative laws, ($\mathfrak{M}$III) B-absorption law, ($\mathfrak{M}$IV) Commutative laws and ($\mathfrak{M}$V)*

(a) Let $a, b \in A$ be such that $F_\vee(a, b) \neq \varnothing$, and let $x, y \in F_\vee(a, b)$ be such that $F_\vee(x, y) \neq \varnothing$. If $x \neq y$, then $z \neq x, y$ for each $z \in F_\vee(x, y)$.
(b) Let $a, b \in A$ be such that $F_\wedge(a, b) \neq \varnothing$, and let $x, y \in F_\wedge(a, b)$ be such that $F_\wedge(x, y) \neq \varnothing$. If $x \neq y$, then $z \neq x, y$ for each $z \in F_\wedge(x, y)$.

Definition 6.2 (Hansen[8]) *An* **algebraic multilattice**, *$(A, F_\vee, F_\wedge)$, is a set A with two operators $F_\vee : A \times A \to 2^A$ and $F_\wedge : A \times A \to 2^A$, that verify the following axioms: (AI) Idempotency laws, (AII) Commutative laws, $(AIII)$ Absorption law, (AIV) H-associative laws and (AV)*

Let $a, b \in A$ be such that $F_\vee(a, b) \neq \varnothing$ (resp. $F_\wedge(a, b) \neq \varnothing$). Then for each $x, y \in F_\vee(a, b)$, $F_\wedge(x, y) \neq \varnothing$ and $x \in F_\vee(a, F_\wedge(x, y))$. (resp. for each $x, y \in F_\wedge(a, b)$, $F_\vee(x, y) \neq \varnothing$ and $x \in F_\wedge(a, F_\vee(x, y))$).

Notice that in our definition 5.6 of an algebraic multilattice, the transitivity of the relation "$a \geq b$ if and only if $F_\vee(a, b) = \{a\}$" is an immediate consequence of the weak associativity of $F_\vee$. This has allowed us to define algebraic multisemilattices, since no property on $F_\wedge$ is required.

In the definition of Benado [1], it is necessary to use the B-absorption law ($\mathfrak{M}$III), the commutative laws ($\mathfrak{M}$IV), and the law ($\mathfrak{M}$V) to guarantee the transitivity and, therefore, the operators $F_\vee$ and $F_\wedge$ are required.

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