Operators on Lorentz Sequence Space II

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1. Introduction

The classical inequalities of Hardy, Capson and Hilbert [HLP] describe the norms of certain matrix operators on the sequence space ℓ_p. In the present paper, we address the problem of finding the norms of these operators when ℓ_p is replaced by the Lorentz sequence space $d(w,1)$ determined by a weighting sequence (w_n).

First we establish the conditions under which the norm of an operator on $d(w,1)$ is determined by decreasing, non-negative sequences. For operators satisfying these conditions, a pleasantly simple general theorem is available. In fact, our problem reduce to the study of a certain sequence, as follows. Let the operator have matrix $(b_{i,j})$, and let $u_j = \sum_{i=1}^{\infty} w_i b_{i,j}$. Write $W_n = w_1 + \ldots + w_n$ (and U_n similarly). Then the norm of B is the supremum of $\frac{U_n}{W_n}$. This amount to saying that is determined by elements of the form $(1, \ldots, 1, 0, \ldots)$.

However, evaluating these quantities in particular cases can be far from trivial, and we turn to the problem of doing so for the classical operators mentioned above. The problem is only complete when we have chosen the weighting sequence, and we consider two natural choices, defined respectively by $w_n = \frac{1}{n^p}$ and $W_n = n^{1-p}$. Each operator then presents the problem of evaluating specific suprema, usually concerning partial sums or tails of series.

Typical results are as follows. For $w_n = \frac{1}{n^p}$, the Cesaro operator has norm $\xi(1+p)$. By contrast, for $W_n = n^{1-p}$, the norm is $\frac{1}{p}$. For the Copson operator, the norm is $\frac{1}{1-p}$ when $w_n = \frac{1}{n^p}$ or $W_n = n^{1-p}$. For the Hilbert operator, with $w_n = \frac{1}{n^p}$, the norm is $\frac{\pi}{\sin p\pi}$ (which is analogous to the ℓ_p case).

In certain cases, the bounds of $(\frac{U_n}{W_n})$ coincide with those of $(\frac{u_n}{w_n})$. Though the sequences in question are often increasing or decreasing, it can be substantially harder to prove this fact than t o show directly that the limit (or the first term) is the supremum. Also, small changes to the operator, or to (w_n), are enough to change $(\frac{U_n}{W_n})$ from an increasing sequence to a decreasing one, or (worse) to one that decrease first, then increase, with obvious implications for the supremum.

The case of $d(w,p)$, with $p > 1$ presents substantial additional features. Some results for this case are given in [Lash] and [JL].

2. General Matrix Operators

For a sequence $x = (x_n)$, we define $|x|$ and the relation $x \leq y$ in the obvious pointwise way. We denote the e_j the sequence having 1 in place j and 0 elsewhere. Let $w = (w_n)$ be a decreasing, non-negative sequence with $\lim_{n\to\infty} w_n = 0$ and $\sum_{n=1}^{\infty} w_n$ divergent. Write $W_n = w_1 + w_2 + \ldots + w_n$. The Lorentz sequence space $d(w,1)$ is the space of sequence x with

$$\|x\|_{w,1} = \sum_{n=1}^{\infty} w_n x_n^*$$

finite, where (x_n^*) is the decreasing rearrangement of $|x_n|$. For such x, one has $W_n x_n^* \to 0$ as $n \to \infty$, and hence by Abel summation

$$\|x\|_{w,1} = \sum_{n=1}^{\infty} W_n (x_n^* - x_{n+1}^*).$$

By Abel summation, this equals

$$\sum_{n=1}^{\infty} (w_n - w_{n+1}) X_n^*.$$

where $X_n^* = x_1^* + x_2^* + \ldots + x_n^*$ (this is where we need the condition $w_n \to 0$). Hence if $X_n^* \leq Y_n^*$ for all n, then $\|x\|_{w,1} \leq \|y\|_{w,1}$. (By Ky fan's Lemma [GK, III.3.1], the same is actually true for symmetric Banach sequence spaces generally.)

Now consider the operator B defined by $Bx = y$, where $y_i = \sum_{j=1}^{\infty} b_{i,j} x_j$. We denote by $\|B\|_{w,1}$, the norm of B as an operator from $d(w,1)$ into itself. We assume throughout that
(1) $b_{i,j} \geq 0$ for all i, j.

This implies that $|B(x)| \leq B(|x|)$ for all x, and hence the non-negative sequence x are sufficient to determine $\|B\|_{w,1}$. A much more delicate problem is to find conditions under which the norm is determined by decreasing sequence x. The next result gives a theoretical answer to this question. However, for the particular operators considered below, the required property is very easily seen directly, whithout this result.

Proposition 1: Suppose that (1) holds, and that
(2) for all subsets M, N of $\mathbb{N}$ having m, n elements respectively, we have

$$\sum_{i \in M} \sum_{j \in N} b_{i,j} \leq \sum_{i=1}^{m} \sum_{j=1}^{n} b_{i,j}.$$

Then $\|B(x)\|_{w,1} \leq \|B(x^*)\|_{w,1}$ for all non-negative elements x of $d(w,1)$. Hence decreasing, non-negative elements are sufficient to detemine $\|B\|_{w,1}$.

Proof. Let $y = Bx$, $z = Bx^*$. We show that

$$y_1^* + \ldots + y_m^* \leq z_1 + \ldots + z_m. \qquad (\forall m)$$

As remarked above, it follows (in any symmetric Banach sequence space) that $\|y\| \leq \|z\|$. (For this, we do not need to know that z_j are in decreasing order, though we shall see that this is in fact implied by (2)). Let $y_i^* = y_{\sigma(i)}$ and let $M = \{\sigma(i) : 1 \leq i \leq m\}$. Also, let $x_j^* = x_{\tau(j)}$. Then

$$\sum_{i=1}^{m} y_i^* = \sum_{i \in m} g_i = \sum_{i \in m} \sum_{j=1}^{\infty} b_{i,\tau(j)} x_j^* = \sum_{j=1}^{\infty} \left(\sum_{i \in m} b_{i,\tau(j)} \right) x_j^*.$$

By Abel summation (since $x_n^* \to 0$), this equals

$$\sum_{n=1}^{\infty} \left(\sum_{i \in m} \sum_{j \in N(n)} b_{i,j} \right) (x_n^* - x_{n+1}^*),$$

where $N(n) = \{\tau(j) : 1 \leq j \leq n\}$. Meanwhile,

$$\sum_{i=1}^{m} z_i = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{i,j} x_j^* = \sum_{n=1}^{\infty} \left(\sum_{i=1}^{m} \sum_{j=1}^{n} b_{i,j} \right) (x_n^* - x_{n+1}^*),$$

the required inequality follows from (2).

Note that condition (2) implies, in particular, that $b_{i,j} \leq b_{1,1}$ for all i, j. However, matrices satisfying condition (2) are by no means instantly recognisable. The next result provides sufficient conditions that are transparently satisfied in many cases of interest, including those considered below. Write

$$r_{i,n} = \sum_{j=1}^{n} b_{i,j} \quad , \quad c_{m,j} = \sum_{i=1}^{m} b_{i,j},$$

the partial sums along row i and column j respectively. Consider the following conditions:
(3) $r_{i,n}$ decreases with i for each n.
(3*) $b_{i,j}$ decreases with i for each j.
(4) $c_{m,j}$ decreases with j for each m.
(4*) $b_{i,j}$ decreases with j for each i.
Clearly, (3*) is stronger than (3), and (4*) is stronger than (4).

Proposition 2: Condition (2) implies (3) and (4). Conversely (3) and (4*), or (4) and (3*), imply (2).

Proof. Suppose that (3) is false, so that $r_{m,n} < r_{m+1,n}$ for some m, n. Let $M = \{1, 2, \ldots, m-1, m+1\}$, $N = \{1, 2, \ldots, n\}$. Then

$$\sum_{i \in M} \sum_{j \in N} b_{i,j} = \sum_{i \in M} r_{i,n} > \sum_{i=1}^{m} r_{i,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{i,j},$$

so (2) fails. Similarly for (4).

Now assume that (3) and (4*) hold, and consider M, N as in (2). For fixed i, the largest n terms $b_{i,j}$ are the first n terms, so

$$\sum_{j \in N} b_{i,j} \leq \sum_{j=1}^{n} b_{i,j} = r_{i,n}.$$

In the same way, by (3),

$$\sum_{i\in M} r_{i,n} \le \sum_{i=1}^{m} r_{i,n} = \sum_{i=1}^{m}\sum_{j=1}^{n} b_{i,j}.$$

Note. A diagonal matrix, decreasing along the diagonal, satisfies (2) but not (3*) or (4*). A matrix that satisfies (3) and (4), but not (2), is

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

For this matrix, if $x = e_3$, then $x^* = e_1$, and (with the above notation) $y_1^* = 2$ while $z_1 = 1$.

Condition (3*) clearly implies that $B(x)$ is decreasing for any non-negative x, while (3) implies that $B(x)$ is decreasing for decreasing, non-negative x, since by Abel summation again,

$$y_i = \sum_{j=1}^{\infty} r_{i,j}(x_j - x_{j+1}).$$

(Hence the z_j in Proposition 1 are decreasing.)

We need one more condition ensuring that at least finite sequences are mapped into $d(w,1)$. Now $b_{i,1} = y_i$, where $y = B(e_1)$. Assuming (3), this decreases with i, so the following condition is equivalent to $B(e_1)$ being in the space $d(w,1)$:

(5) $\sum_{i=1}^{\infty} w_i b_{i,1}$ is convergent.

By Abel summation, this series can be rewritten $\sum_{m=1}^{\infty} c_{m,1}(w_m - w_{m+1})$. Given (4), the same is true with $c_{m,j}$ replacing $c_{m,1}$, so that $\sum_{i=1}^{\infty} w_i b_{i,j}$ is convergent for each j. We define

$$u_j = u_j(B,w) = \sum_{i=1}^{\infty} w_i b_{i,j}.$$

Formally, (u_j) is the sequence $B^*(w)$. Condition (4) implies that it is decreasing. Note that $u_1 = \|B(e_1)\|_{w,1}$, but for $j > 1$, u_j need not equal $\|B(e_j)\|_{w,1}$ unless (3*) holds. The correct interpretation of u_j (or rather U_j) emerges in the proof of the next result, our basic theorem on general matrix operators.

Theorem 3. Suppose that B satisfies conditions (1), (2), (5). Let $u_j = \sum_{i=1}^{\infty} w_i b_{i,j}$ and $U_n = u_1 + \ldots + u_n$. Then B is a bounded operator on $d(w,1)$ if and only if $(\frac{U_n}{W_n})$ is bounded above, and

$$\|B\|_{w,1} = \sup_{n\ge 1} \frac{U_n}{W_n}.$$

This norm can be evaluated by considering only elements of the form $e_1 + \ldots + e_n$.

Proof: Let (x_j) be a decreasing, non-negative sequence. Then (y_i) is also decreasing, so

$$\begin{aligned} \|Bx\|_{w,1} &= \sum_{i=1}^{\infty} w_i y_i \\ &= \sum_{i=1}^{\infty} w_i \sum_{j=1}^{\infty} b_{i,j} x_j \\ &= \sum_{j=1}^{\infty} u_j x_j \\ &= \sum_{j=1}^{\infty} U_j (x_j - x_{j+1}), \end{aligned}$$

while

$$\|x\|_{w,1} = \sum_{j=1}^{\infty} W_j (x_j - x_{j+1}).$$

Let $M = \sup_{n\ge 1} \frac{U_n}{W_n}$. Then, clearly,

$$\|Bx\|_{w,1} \le M \|x\|_{w,1}.$$

Further, if $x = e_1 + \ldots + e_n$, then $\|x\|_{w,1} = W_n$ and $\|Bx\|_{w,1} = U_n$, so such elements suffice to show that $\|B\|_{w,1} = M$.

In certain cases, it is enough to consider the sequence $(\frac{u_n}{w_n})$ instead of $(\frac{U_n}{W_n})$, because of the well-known facts listed in the following lemma.

Lemma 1. (i) If $m \le \frac{u_n}{w_n} \le M$ for all n, then $m \le \frac{U_n}{W_n} \le M$ for all n.

(ii) If $(\frac{u_n}{w_n})$ is increasing (or decreasing), then so is $(\frac{U_n}{W_n})$.

(iii) If $\frac{u_n}{w_n} \to M$ as $n \to \infty$, then $\frac{U_n}{W_n} \to M$ as $n \to \infty$ (also with $L = \infty$).

Proof: Elementary.

Hence, for example, if $(\frac{u_n}{w_n})$ is increasing and tends to the limit M, then $\|B\|_{w,1} = M$. The same conclusion holds provided that we can show that $\frac{u_1}{w_1} \le \frac{u_n}{w_n} \le M$ for all n. We shall see that is some cases, this is much easier than showing that the sequence is increasing.

3. Partial Sums and Tails of $\sum \frac{1}{n^p}$

The following mostly well-known facts will be used repeatedly in evaluating the suprema and infima arising in our chosen particular cases. Let $p > 0$, and write

$$x_n = \frac{1}{n^p},$$
$$y_n = \int_{n-1}^{n} \frac{1}{t^p} dt,$$

and as usual $X_n = x_1 + \ldots + x_n$, etc. For $p < 1$, the usual integral comparison gives

$$y_2 + \ldots + y_n \le X_n \le Y_n,$$

or

$$\frac{1}{1-p}(n^{1-p} - 1) \le X_n \le \frac{n^{1-p}}{1-p},$$

hence $\frac{X_n}{Y_n} \to 1$ as $n \to \infty$. We need to know also that $\frac{X_n}{Y_n}$ is increasing. The following is the key lemma.

Lemma 2: With y_n as above (for any $p > 0$), $n^p y_n$ decreases with n and $n^p y_{n+1}$ increases with n.

Proof: Write $s_n = n^p y_n$. Then

$$s_{n+1} = (n+1)^p \int_{n}^{n+1} \frac{1}{t^p} dt = (n+1)^p \int_{n-1}^{n} \frac{dt}{(t+1)^p}.$$

For $n - 1 \le t \le n$, we have $\frac{(n+1)}{n} \le \frac{(t+1)}{t}$, hence $\frac{(n+1)^p}{(t+1)^p} \le \frac{n^p}{t^p}$. Hence $s_{n+1} \le s_n$. Similarly for the second statement.

Proposition 4. Let $0 < p < 1$ and let $X_n = \sum_{j=1}^{n} \frac{1}{j^p}$. Then $\frac{X_n}{n^{1-p}}$ increases and tends to $\frac{1}{1-p}$.

Proof: By Lemma 2, $\frac{x_n}{y_n}$ increases. Hence, by Lemma 1 (ii), $\frac{X_n}{Y_n}$ increases. The limit follows from the inequalities above.

We now consider the tail of the series for $\xi(1+p)$. For the tail of a series, the analogous result to Lemma 1 (ii) is the following.

Lemma 3: Suppose that $x_n > 0$, $y_n > 0$ for all n and that $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are convergent. Let $X_{(n)} = \sum_{j=n}^{\infty} x_j$, similarly $Y_{(n)}$. If $(\frac{x_n}{y_n})$ is increasing (or decreasing), then so is $(\frac{X_{(n)}}{Y_{(n)}})$.

Proof: Elementary.

Proposition 5. Let $p > 0$ and let $X_{(n)} = \sum_{k=n}^{\infty} \frac{1}{k^{1+p}}$. Then $n^p X_{(n)}$ is decreasing, $(n-1)^p X_{(n)}$ increasing. Both tend to $\frac{1}{p}$ as $n \to \infty$.

Proof: Let $x_n = \frac{1}{n^{1+p}}$ and

$$y_n = \int_{n-1}^{n} \frac{1}{t^{1+p}} dt.$$

Then $Y_{(n+1)} = \frac{1}{pn^p}$. By the usual integral comparison,

$$\frac{1}{pn^p} \le X_{(n)} \le \frac{1}{p(n-1)^p},$$

which implies the stated limits. By Lemma 2, $(\frac{x_n}{y_{n+1}})$ is decreasing, so by Lemma 3, $\frac{X_{(n)}}{Y_{(n+1)}} = pn^p X_{(n)}$ is decreasing. Similarly, $\left(\frac{X_{(n)}}{Y_{(n)}}\right)$ is increasing.

Remark. This is stated without proof in [Benn 2], Remark 4.10.

4. The Cesaro Operator and Its Transpose (Copson Operator)

The Cesaro operator A is dedined by $y = Ax$, where

$$y_n = \frac{1}{n}(x_1 + x_2 + \ldots + x_n).$$

It is given by the Cesaro matrix:

$$a_{i,j} = \begin{cases} \frac{1}{i} & \text{for } \; j \le i \\ 0 & \text{for } \; j > i \end{cases}.$$

This is a lower triangular matrix. In our terms, it satisfies conditions (3) and (4*). When A is regarded as an operator on ℓ_p (where $p > 1$), Hardy's inequality (see e.g. [HLP], [Benn 1] and [LD]) states that $\|A\|_p = \frac{p}{p-1}$. (The element e_1 is enough to show that A does not map ℓ_1 into ℓ_1.)

Condition (50 requires convergence of $\sum_{k=1}^{\infty} \frac{w_n}{k}$, and u_n is given by

$$u_n = \sum_{k=n}^{\infty} \frac{w_k}{k}.$$

For the weighting sequence $w_n = \frac{1}{n^p}$, our earlier results provid an immediate solution to our problem.

Theorem 6: Let A be the Cesaro operator, and let $w_n = \frac{1}{n^p}$, where $0 < p < 1$. Then

$$\|A\|_{w,1} = \xi(1+p).$$

Proof: We now have $\frac{w_k}{k} = \frac{1}{k^{1+p}}$, so $u_n = X_{(n)}$ in the notation of Proposition 5, which tells us that $n^p u_n$ $(= \frac{u_n}{w_n})$ is decreasing and tends to $\frac{1}{p}$. By Lemma 1, it follows that

$$\|A\|_{w,1} = \frac{u_1}{w_1} = \xi(1+p).$$

The Capson operator C is defined by $y = Cx$, where

$$y_n = \sum_{k=n}^{\infty} \frac{x_k}{k}.$$

It is given by the transpose of the matrix of the Cesaro operator:

$$c_{i,j} = \begin{cases} \frac{1}{j} & \text{for } i \leq j \\ 0 & \text{for } i > j \end{cases}.$$

This is an upper triangular matrix satisfying (4) and (3*). The classical inequality of Copson [Cop] states that $\|C\|_p = p$ as an operator on ℓ_p.

A pleasantly simple statements can be made about the norm of C for general (w_n). With the notation of section 2,

$$u_n = \frac{1}{n}(w_1 + \ldots + w_n) = \frac{W_n}{n}.$$

Following [R], we define the 1-regularity constant of (w_n) to be

$$r_1(w) = \sup_{n \geq 1} \frac{W_n}{n w_n}$$

and say that $w = (w_n)$ is 1-regular if this is finite.

Proposition 7. If $w = (w_n)$ is 1-regular, then C maps $d(w,1)$ into itself. Also, we have

$$\|C\|_{w,1} \leq r_1(w).$$

Proof. Since

$$u_n = \frac{W_n}{n} \leq r_1(w) w_n \qquad (\forall n),$$

then by Theorem 3 and Lemma 1 (i), it follows that $\|C\|_{w,1} \leq r_1(w)$.

Proposition 8. If

$$\sup \frac{1}{W_n} \sum_{k=1}^{n} \frac{W_k}{k} < \infty$$

then the Capson operator C is a bounded operator from $d(w,1)$ into itself. Also, we have

$$\|C\|_{w,1} = \sup_{n \geq 1} \frac{1}{W_n} \sum_{k=1}^{n} \frac{W_k}{k}.$$

Proof: Since

$$u_n = \sum_{j=1}^{\infty} c_{j,n} w_j = \frac{1}{n}(w_1 + \ldots + w_n) = \frac{W_n}{n},$$

then, by hypothesis and Theorem 3, it follows that

$$\|C\|_{w,1} = \sup \frac{1}{W_n} \sum_{k=1}^{n} \frac{W_k}{k}.$$

Theorem 9. Let C be the Capson operator, and let $w_n = \frac{1}{n^p}$, where $0 < p < 1$. Then

$$\|C\|_{w,1} = \frac{1}{1-p}.$$

Proof: With our standing notation,

$$\frac{u_n}{w_n} = \frac{W_n}{n w_n} = \frac{W_n}{n^{1-p}}.$$

Our W_n is the X_n of Proposition 4, which tells us that $\frac{W_n}{n^{1-p}}$ increases and tends to $\frac{1}{1-p}$. The statement follows by (ii) and (iii) of Lemma 1. (Of course, this also shows that $r_1(w) = \dfrac{1}{1-p}$).

Remark. When $p = 1$, so that $w_n = \dfrac{1}{n}$, we have

$$\frac{u_n}{w_n} = W_n \to \infty \quad \text{as} \quad n \to \infty,$$

so C is not a bounded operator on $d(w,1)$, although of course it satisfies condition (5).

5. The Hilbert Operator

Two versions of the Hilbert operator, which we denote by H_1 and H_0 respectively, are given by the matrices

$$h_{i,j} = \frac{1}{i+j} \qquad , \qquad h_{i,j} = \frac{1}{i+j-1}.$$

These are Hankel matrices satisfying (3*) and (4*). Hilbert's inequality (see e.g. [HLP]) gives the norm of both operators on ℓ_p (for $p > 1$) as $\pi/\sin(\dfrac{\pi}{p})$.

We start by considering H_1, with $w_n = \dfrac{1}{n^p}$. In our usual notation, we have

$$u_n = \sum_{i=1}^{\infty} \frac{1}{i^p(i+n)}.$$

Theorem 10. With u_n defined in this way, we have $\sup\limits_{n\geq 1} n^p u_n = \dfrac{\pi}{\sin p\pi}$. Hence if $w_n = \dfrac{1}{n^p}$, where $0 < p < 1$, then

$$\|H_1\|_{w,1} = \frac{\pi}{\sin p\pi}.$$

Proof: By Comparison with the well-known integral

$$\int_0^{\infty} \frac{dt}{t^p(t+c)} = \frac{\pi}{c^p \sin p\pi} \qquad (0 < p < 1),$$

we have $u_n \leq \dfrac{\pi}{n^p \sin p\pi}$, hence

$$n^p u_n \leq \frac{\pi}{\sin p\pi}.$$

Also, we have

$$u_n = \int_0^{\infty} \frac{dt}{t^p(t+n)},$$

and

$$\int_0^1 \frac{dt}{t^p(t+n)} \leq \int_0^1 \frac{dt}{nt^p} = \frac{1}{(1-p)n}.$$

Hence

$$n^p u_n \geq \frac{\pi}{\sin p\pi} - \frac{1}{(1-p)n^{1-p}},$$

which proves the stated supremum. Then by Lemma 1, we have

$$\|H\|_{w,1} = \frac{\pi}{\sin p\pi}.$$

Remark 1. When $p = 1$, we have

$$u_n = \sum_{i=1}^{\infty} \frac{1}{i(i+n)} = \frac{1}{n}(1 + \frac{1}{2} + \ldots + \frac{1}{n}),$$

hence $nu_n \to \infty$ as $n \to \infty$, and H_1 is not a bounded operator on $d(w,1)$.

Remark 2. The operator H_0 (with $w_n = \dfrac{1}{n^p}$) is much harder to deal with. Clearly, $u_n(H_0,w) = u_{n-1}(H_1,w)$ for $n \geq 2$, and $u_1(H_0,w) = \xi(1+p)$. The limit of $n^p u_n$ is still $\dfrac{\pi}{\sin p\pi}$, but this less than $\xi(1+p)$ when p is less than approximately 0.32. It is quite easy to show that $n^p u_n \leq \dfrac{\pi}{\sin p\pi}$ for large enough n. Computation indicate that $n^p u_n$ either increase throughout, or decreases for a certain number of terms and then increases. This, if proved, would imply that $\|H_0\|_{w,1}$ is the greater of $\xi(1+p)$ and $\dfrac{\pi}{\sin p\pi}$.

We turn to the case where w_n is defined by $W_n = n^{1-p}$ (where $0 < p < 1$, so that

$$w_n = n^{1-p} - (n-1)^{1-p} = \int_{n-1}^{n} \frac{1-p}{t^p} dt.$$

Note first that, with the notation of section 2,

$$U_n = \sum_{i=1}^{\infty} r_{i,n} w_i = \sum_{i=1}^{\infty} W_i (r_{i,n} - r_{i+1,n}).$$

This time, we consider H_0 first, since it turns out (in the same way as in Theorem 9) that we have solved the problem for this operator already! For H_0, we have:

Theorem 11. With w_n defined by $W_n = n^{1-p}$, we have

$$\|H_0\|_{w,1} = \frac{\pi}{\sin p\pi}.$$

Proof: We have

$$r_{i,n} = \frac{1}{i} + \ldots + \frac{1}{i+n-1},$$

hence

$$r_{i,n} - r_{i+1,n} = \frac{1}{i} - \frac{1}{i+n} = \frac{n}{i(i+n)},$$

and by the above

$$\frac{U_n}{W_n} = \frac{1}{n^{1-p}} \sum_{k=1}^{\infty} k^{1-p} \frac{n}{k(k+n)} = n^p \sum_{k=1}^{\infty} \frac{1}{k^p(k+n)}.$$

This is previsly the $\frac{u_n}{w_n}$ of Theorem 10, so we have

$$\|H_0\|_{w,1} = \frac{\pi}{\sin p\pi}.$$

For H_1, we have instead

$$r_{i,n} - r_{i+1,n} = \frac{1}{i+1} - \frac{1}{i+n+1} = \frac{n}{(i+1)(i+n+1)},$$

so that

$$\frac{U_n}{W_n} = n^p \sum_{i=1}^{\infty} \frac{i^{1-p}}{(i+1)(i+n+1)}.$$

Theorem 12. With w_n defined by $W_n = n^{1-p}$, we have

$$\|H_0\|_{w,1} = \frac{\pi}{\sin p\pi}.$$

Proof: The norm estimation only requires slight adaptations to the proof of Theorem 10. Clearly,

$$\frac{U_n}{W_n} \le n^p \frac{1}{i^p(i+n)}.$$

As seen in Theorem 10, this is not greater that $\frac{\pi}{\sin p\pi}$. For an $N \ge 2$,

$$\begin{aligned}\sum_{i=N-1}^{\infty} \frac{i^{1-p}}{(i+1)(i+n+1)} &\ge (\frac{N-1}{N})^{1-p} \sum_{i=N-1}^{\infty} \frac{(i+1)^{1-p}}{(i+1)(i+n+1)} \\ &\ge (\frac{N-1}{N})^{1-p} \sum_{i=N}^{\infty} \frac{1}{i^p(i+n)}.\end{aligned}$$

As in Theorem 10, we see that

$$n^p \int_N^{\infty} \frac{1}{t^p(t+n)} dt \to \frac{\pi}{\sin p\pi} \quad \text{as} \quad n \to \infty,$$

from which it follows that $\|H_1\|_{w,1} = \frac{\pi}{\sin p\pi}$.

References

[Benn 1]G. Bennett, Lower bounds for matrices, Linear Algebra and Appl. 82(1986), 81-98.

[Benn 2] G. Bennett, Factorizing the Classical Inequalities, Mem. Amer. Math. Soc. (1996).

[Cop]E. T. Copson, Notes on series of positive terms, J. London Math. Soc. 2(1927), 9-12.

[GK] I.C. Gokhberg and M.G. Krien, Introduction to the Theory of Linear Nonselfadjoint Operators, Amer. Math. Soc., (1969).

[HLP] G.H. Hardy, J. Littlewood and G. Polya, Inequalities, Cambreidge Univ. Press, (1934).

[JL] G.J.O. Jameson and R. Lashkaripour, Norms of operators on the Lorentz sequence space $d(w,p)$, preprint.

[Lash] R. Lashkaripour, Lower bounds and norms of operators on Lorentz sequence spaces, Doctoral dissertation (Lancaster, 1997).

[LD]R. Lashkaripour, D. Dehhabeh, Lower bounds of operators on sequence spaces, Proceeding of the 11th Seminar on Mathematical Analysis and its applications(2001), to appear.

[R] S. Reisner, A factorization theorem in Banach lattices and its application to Lorentz spaces, Ann. Inst. Fourier, 31 (1981), 239-255.