The Strong Law of Large Numbers for Pairwise Negatively Dependent Random Variables

H.R.NILI SANI, A.BOZORGNIA
Department of Statistics
Faculty of Mathematical Sciences
Ferdowsi University
MASHHAD, IRAN

Abstract:-In many stochastic models, the assumption of independent among the random variables (henceforth r.v.s) is not plausible. In fact, increase in some r.v.s are often related to decrease in other r.v.s and the assumption of pairwise negative dependent is more appropriate than independent assumption. In this paper strong laws of large numbers (SLLN) are obtained for sum $\sum X_n$ under certain condition where $\{X_n, n \geq 1\}$ is a sequence of pairwise negatively dependent r.v.s.

Key-words: -Strong law of large numbers, Pairwise negatively dependent, Cesaro uniformly integrable.

1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of integrable r.v.s defined on the same probability space and put $S_n = \sum_{i=1}^{n} X_i$, $\overline{X}_n = \frac{S_n}{n}$. Chandra (1992), under certain conditions, modified the SLLN of Kolmogrov (Th.5.4.2 of Chung (1974)) and the SLLN of Lander and Rogge (1986) for pairwise independent r.v.s which are not necessarily identically distributed and satisfy certain moment conditions. Bozorgnia, Patterson and Taylor (1996) obtained the SLLN for sums of an array rowwise negatively dependent r.v.s under certain conditions. Amini (2000) has proved the SLLN for special negatively dependent r.v.s (Theorem 5.5) and for weighted sums of uniformly bounded negatively dependent r.v.s (Theorem 3.7). He has also proved the WLLN for special pairwise negatively dependent r.v.s (Theorems 5.6,5.7,5.8). In this paper, we modified the theorems of SLLN Chandra (1992) for pairwise negatively dependent r.v.s that satisfy certain moment conditions.

Definition 1 The random variables $X_1, \cdots, X_n$ $(n \geq 2)$ are said to be pairwise negatively dependent (henceforth pairwise *ND*) if the following inequality holds,

$$P(X_i > x_i, X_j > x_j) \leq P(X_i > x_i) P(X_j > x_j) \quad (1)$$

for all $x_i, x_j \in \Re^1$, $i \neq j$. It can be shown that (1) is equivalent to

$$P(X_i \leq x_i, X_j \leq x_j) \leq P(X_i \leq x_i) P(X_j \leq x_j) \quad (2)$$

for all $x_i, x_j \in \Re^1$, $i \neq j$.

Definition 2 The random variables $X_1, \cdots, X_n$ $(n \geq 2)$ are said to be negatively associated (*NA* for short) if for every pair of disjoint nonempty subsets A_1, A_2 of $\{1, ..., n\}$,

$$Cov(f_1(X_i, i \in A_1), f_2(X_i, i \in A_2)) \leq 0 \quad (3)$$

whenever f_1 and f_2 are monotone in the same direction in each coordinate, the remaining $n-1$ kept fixed and such that $E(f_1^2(X_i, i \in A_1)) < \infty$ and $E(f_2^2(X_i, i \in A_2)) < \infty$.

Theorem 1 (Amini (2000)) Random variables X_1, X_2 are *ND* if and only if for every function f and g that are monotone in the same direction,

$$Cov(f(X_1), g(X_2)) \leq 0 .$$

Corollary 1 If $X_1, \cdots, X_n (n \geq 2)$ are *NA* r.v.s then they are pairwise negatively dependent r.v.s.

Definition 3 A sequence $\{X_n, n\geq 1\}$ of r.v.s is said to be *Cesaro uniformly integrable* if

$$\lim_{N\to\infty} \sup_n [n^{-1}\sum_{i=1}^{n} E(|X_i| I(|X_i| > N)] = 0 \qquad (4)$$

Definition 4 A sequence $\{a_n, n\geq 1\}$ of non-negative reals is said to be *Cesaro bounded* if the sequence $\{n^{-1}(a_1+\cdots+a_n)\}$ is bounded.

2 Strong Convergence

In this paper C stand for a generic constant not necessarily the same at appearance. Also $\{f(n)\}$ will stand for an increasing sequence such that $f(n)>0$ for each n and $f(n)\to\infty$. The next theorem can be obtained from the argument of Csorgo et al (1983).

Theorem 2 Let $\{X_n, n\geq 1\}$ be a sequence of non-negative r.v.s with finite $Var(X_n)$. Assume that

(i) $\sup_{n\geq 1}\left[\sum_{k=1}^{n} E(X_k)/f(n)\right] = A(say) < \infty$;

(ii) there is a double sequence $\{r_{ij}\}$ of non-negative real such that

$$Var(S_n) \leq \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij} \quad \text{for each } n\geq 1;$$

(iii) $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} r_{ij}/(f(i\vee j))^2 < \infty$, $i\vee j = \max(i,j)$.

Then $[S(n)-E(S(n))]/f(n)\to 0$ almost surely as $n\to\infty$.

The proof of Theorem 2 can be found in Chandra (1992).

Proposition 1 Let $\{X_n, n\geq 1\}$ be a sequence of pairwise *ND* random variables. If $\{f_n, n\geq 1\}$ be a sequence of Borel functions all of which are monotone increasing (or all monotone decreasing) then $\{f_n(X_n), n\geq 1\}$ is a sequence of pirwise *ND* random variables.

The proof of Proposition 1 can be found in Amini (2000).

Corollary 2 Let $\{X_n, n\geq 1\}$ be a sequence of pairwise *ND* random variables. Then $\{X_n^+, n\geq 1\}$ and $\{X_n^-, n\geq 1\}$ are two sequences of pairwise *ND* random variables where X_n^+ and X_n^- are positive and negative parts of random variable X_n respectively.

Theorem 3 Let $\{X_n, n\geq 1\}$ be a sequence of pairwise *ND* r.v.s with finite $Var(X_n)$. Assume that

(i) $\sup_{n\geq 1}\left[\sum_{k=1}^{n} E(|X_k - E(X_k)|/f(n)\right] < \infty$,

and

(ii) $\sum_{n=1}^{\infty} (f(n))^{-2} Var(X_n) < \infty$,

Then $[S(n)-E(S(n))]/f(n)\to 0$ almost surely as $n\to\infty$.

Proof Put $Y_n = (X_n - E(X_n))^+$ and $Z_n = (X_n - E(X_n))^-$ ($n\geq 1$). It is sufficient to show that as $n\to\infty$,

$$(f(n))^{-1}\sum_{i=1}^{n}(Y_i - E(Y_i)) \to 0 \text{ a.s.},$$

and

$$(f(n))^{-1}\sum_{i=1}^{n}(Z_i - E(Z_i)) \to 0 \text{ a.s.} \qquad (5)$$

Since $Var(Y_n) \leq E(Y_n^2) \leq Var(X_n)$ and $E(Y_n)\leq E|X_n - E(X_n)|$ ($n\geq 1$), it follows that condition (i) of Theorem 2 is valid for $\{Y_n\}$. Similarly, it is valid for $\{Z_n\}$. Under pairwise *ND* condition we have

$$Var(\sum_{i=1}^{n} Y_i) \leq \sum_{i=1}^{n} Var(Y_i) \leq \sum_{i=1}^{n} Var(X_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij}$$

$\forall\ n\geq 1$, where $r_{ii} = Var(X_i)$ and $r_{ij} = 0$ for each $i\neq j$. It follows from Theorem 2 that

$$\frac{1}{f(n)}\sum_{i=1}^{n}(Y_i - E(Y_i)) \to 0 \quad \text{almost surely.}$$

Replacing X_n by $W_n = -X_n$ and $Z_n = (X_n - E(X_n))^-$ by $Z_n = (W_n - E(W_n))^+$ one gets the second part of (5). Since

$$\frac{S_n - E(S_n)}{f(n)} = \frac{\sum_{i=1}^{n}(Y_i - E(Y_i)) - \sum_{i=1}^{n}(Z_i - E(Z_i))}{f(n)} + \frac{(\sum_{i=1}^{n} E(Y_i) - \sum_{i=1}^{n} E(Z_i))}{f(n)}$$

we have

$$\frac{S_n - E(S_n)}{f(n)} \to 0 \quad \text{a.s.}$$

Proposition 2 Let $\{X_n, n \ge 1\}$ be pairwise *ND* r.v.s. Suppose $\{B_n, n \ge 1\}$ is a sequence of semi intervals $(-\infty, x_n]$ ($(-\infty, x_n)$, $[x_n, \infty)$ or (x_n, ∞)), then $\{X_n I(X_n \in B_n), n \ge 1\}$ is a sequence of pairwise *ND* r.v.s.

Theorem 4 Let $\{X_n, n \ge 1\}$ be a sequence of pairwise *ND* integrable random variables such that there is a sequence $\{B_n, n \ge 1\}$ of Borel subsets of $\Re^1$ that are semi intervals as $(-\infty, x_n]$, ($(-\infty, x_n)$, $[x_n, \infty)$ or (x_n, ∞)) satisfying the following conditions (a)-(d):

(a) $\sum_{n=1}^{\infty} P(X_n \in B_n^c) < \infty$;

(b) $\sum_{i=1}^{n} E(X_i I(X_i \in B_i^c)) = o(f(n))$;

(c) $\sum_{n=1}^{\infty} (f(n))^{-2} Var(X_n I(X_n \in B_n)) < \infty$;

and

(d) $\sup_{n\ge1} [\sum_{k=1}^{n} E(|X_k| I(X_k \in B_k)) / f(n)] < \infty$;

here B_n^c is the complement of B_n. Then $[S(n) - E(S(n))] / f(n) \to 0$ almost surely as $n \to \infty$.

Proof Let $Y_n = X_n I(X_n \in B_n)$, $n \ge 1$. By (c) and (d), Theorem 3 applied to $\{Y_n\}$ yields $\frac{1}{f(n)} \sum_{i=1}^{n} (Y_i - E(Y_i)) \to 0$ almost surely as $n \to \infty$. By (b), we get $(f(n))^{-1} \sum_{i=1}^{n} (Y_i - E(X_i)) \to 0$ almost surely as $n \to \infty$. Since r.v.s $\{X_n, n \ge 1\}$ and $\{Y_n, n \ge 1\}$ are equivalent then by (a) and the first Borel-Cantelli lemma, the desired results follows.

The next theorem, our first main result, is an extension of classical Kolmogorov SLLN for pairwise dependent r.v.s (Chandra 1992).

Theorem 5 Let $\{X_n, n \ge 1\}$ be a sequence of pairwise *ND* r.v.s, put $G(x) = \sup_{n\ge1} P(|X_n| \ge x)$ and $h(x) = \sup_{n\ge1} P(X_n \le -x)$ for $x \ge 0$. If $\int_0^{\infty} G(x)dx < \infty$ and $x^{1+r} h(x) \to 0$ as $x \to \infty$ for some $r > 1$, then $\frac{1}{n} \sum_{i=1}^{n} c_i (X_i - E(X_i)) \to 0$ almost surely as $n \to \infty$ for each bounded sequence $\{c_n\}$.

Proof It is suffices to prove the result for $c_n = 1$. To this end, we use Theorem 4 with $B_n = (-\infty, n]$ for all $n \ge 1$. Condition (a) follows since

$$\sum_{n=1}^{\infty} P(X_n \in B_n^c) \le \sum_{n=1}^{\infty} P(|X_n| > n) \le \sum_{n=1}^{\infty} G(n) < \infty .$$

To verify condition (b), note that for any non-negative random variables Z and $a \ge 0$

$$E(ZI(Z \ge a)) = a\, P(Z \ge a) + \int_a^{\infty} P(Z > x)dx .$$

Hence

$$E(X_n I(X_n > n)) \le n P(|X_n| \ge n) + \int_n^{\infty} G(x)dx \to 0 ,$$

so that condition (b) holds. Obviously, condition (d) holds. Thus, it remains to verify condition (c).

$$\sum_{n=1}^{\infty} n^{-2} E(X_n^2 I(X_n \le n)) = \sum_{n=1}^{\infty} n^{-2} \int_0^{\infty} P(X_n^2 I(X_n \le n) > x)dx$$

$$= \sum_{n=1}^{\infty} n^{-2} [\int_0^{n^2} (P(X_n < -\sqrt{x}) + P(\sqrt{x} < X_n \le n))dx$$

$$+ \int_{n^2}^{\infty} P(X_n < -\sqrt{x})dx]$$

we have

$$\sum_{n=1}^{\infty} n^{-2} \int_{n^2}^{\infty} P(X_n < -\sqrt{x})dx \le 2 \sum_{i=1}^{\infty} \sum_{n=1}^{i} n^{-2} \int_i^{i+1} yh(y)dy ,$$

since $y^{1+r} h(y) \to 0$, for every $e > 0$ there exist $M > 0$ such that if $y > M$ then $y^{1+r} h(y) < e$. Hence

$$\sum_{i=1}^{\infty} \sum_{n=1}^{i} n^{-2} \int_i^{i+1} yh(y)dy \le \sum_{i=1}^{M} \sum_{n=1}^{i} n^{-2} \int_i^{i+1} yh(y)dy$$

$$+\sum_{i=M+1}^{\infty}\sum_{n=1}^{i} n^{-2}\int_{i}^{i+1}\frac{y^{1+r}}{y}h(y)dy$$

$$\leq \sum_{i=1}^{M}\sum_{n=1}^{i} n^{-2}\int_{i}^{i+1} yh(y)dy + \sum_{i=M+1}^{\infty}\int_{i}^{i+1} y^{1+r}h(y)dy \frac{\sum_{n=1}^{i} n^{-2}}{i^{r}}$$

$$\leq \sum_{i=1}^{M}\sum_{n=1}^{i} n^{-2}\int_{i}^{i+1} yh(y)dy + \sum_{i=M+1}^{\infty} e\left(\frac{1}{i^{r}}(1+(1-\frac{1}{i}))\right) < \infty ,$$

moreover we have

$$\sum_{n=1}^{\infty} n^{-2}\int_{0}^{n^2}(P(X_n < -\sqrt{x}) + P(\sqrt{x} < X_n \leq n))dx$$

$$\leq \sum_{n=1}^{\infty} n^{-2} \, {}^{n} \; 2 \quad P(X_n \quad - \quad) + P \;\; X \qquad))dy$$

$$\leq \sum_{n=1}^{\infty} n^{-2}\int_{0}^{n} 4yG(y)dy = \sum_{i=1}^{\infty}\sum_{n=i}^{\infty} n^{-2}\int_{i-1}^{i} 4yG(y)dy$$

$$\leq \frac{2p^{2}}{3}\int_{0}^{1} yG(y)dy + \sum_{i=2}^{\infty} 4i\int_{i-1}^{i} G(y)\left(\int_{i-1}^{\infty}\frac{dx}{x^{2}}\right)dy < \infty .$$

Theorem 6 Let $\{X_n, n \geq 1\}$ be pairwise ND r.v.s and $g_n : (0,\infty) \to (0,\infty)$ be such that for each $n \geq 1$ as x increase

$$h_n(x) = \frac{g_n(x)}{x} \uparrow \qquad (6)$$

and $C = \inf_{n\geq 1}\lim_{x\to 0^+} h_n(x) > 0$. Assume $\sum_{n=1}^{\infty} E(g_n^2(|X_n|)) < \infty$. Then $[S(n) - E(S(n))]/f(n) \to 0$ almost surely as $n \to \infty$.

Proof We use Theorem 4 with $B_n = (-\infty, a_n]$, $a_n > 0$. To verify condition (a) note that $\sum_{n=1}^{\infty} P(X_n > a_n) \leq \sum_{n=1}^{\infty} P(g_n(|X_n|) \geq g_n(a_n)) < \infty$.

Next, note that

$$\sum_{n=1}^{\infty} E(X_n I(X_n > a_n)/f(n))$$

$$\leq \frac{1}{f(1)}\sum_{n=1}^{\infty} E\left(\frac{|X_n|}{g_n(|X_n|)} g_n(|X_n|) I(|X_n| > a_n)\right)$$

$$\leq \frac{1}{Cf(1)}\sum_{n=1}^{\infty} E(g_n(|X_n|) I(|X_n| > a_n)) < \infty ,$$

so that condition (b) follows by the Kronecker lemma. Condition (c) follows, since

$$\sum_{n=1}^{\infty}\frac{1}{(f(n))^2} E((X_n^2) I(X_n \leq a_n))$$

$$\leq \frac{1}{C^2 f^2(1)}\sum_{n=1}^{\infty} E(g_n(|X_n|))^2 < \infty .$$

And finally condition ”d” follows, since

$$\sup_{n\geq 1}\sum_{k=1}^{n}(E(|X_k| I(X_k \leq a_k))/f(n))$$

$$\leq \frac{1}{f(1)}\sum_{k=1}^{\infty} E\left(\frac{|X_k|}{g_k(|X_k|)} g_k(|X_k|)\right)$$

$$\leq \frac{1}{Cf(1)}\sum_{k=1}^{\infty} E(g_k(|X_k|)) < \infty .$$

Theorem 7 Let $\{X_n, n \geq 1\}$ be pairwise ND r.v.s and $g_n : (0,\infty) \to (0,\infty)$ be such that for each $n \geq 1$ as x increase

$$h_n(x) = \frac{g_n(x)}{x} \downarrow \qquad (7)$$

$C_0 = \inf_{n\geq 1}\lim_{x\to\infty} h_n(x) > 0$. Then the conclusion of Theorem 6 holds.

Proof The proof of Theorem 6 goes through except that we have to use the upper bound $\frac{1}{C_0}$ for $\frac{|X_n|}{g_n(|X_n|)}$. Then

$$\sum_{n=1}^{\infty} E(X_n I(X_n > a_n)/f(n))$$

$$\leq \frac{1}{C_0 f(1)}\sum_{n=1}^{\infty} E(g_n(|X_n|) I(|X_n| > a_n)) < \infty$$

so that condition (b) follows by Kronecker lemma. Condition (c) follows, since

$$\sum_{n=1}^{\infty}\frac{1}{(f(n))^2} E((X_n^2) I(X_n \leq a_n))$$

$$\leq \frac{1}{C_0^2 f^2(1)}\sum_{n=1}^{\infty} E(g_n(|X_n|))^2 < \infty$$

And finally condition ”d” follows, since

$$\sup_{n\geq 1}\sum_{k=1}^{n}(E(|X_k| I(X_k \leq a_k))/f(n))$$

$$\leq \frac{1}{C_0 f(1)}\sum_{n=1}^{\infty} E(g_n(|X_n|)) < \infty .$$

The next theorem is an analogue of the SLLN of Chung (1947).

Theorem 8 Let $\{X_n, n \geq 1\}$ be pairwise *ND* r.v.s and $g_n : (0,\infty) \to (0,\infty)$ be such that for each $n \geq 1$ as x increase

$$h_n(x) = \frac{g_n(x)}{x} \uparrow, \quad k_n(x) = \frac{g_n(x)}{x^2} \downarrow \qquad (8)$$

$b = \inf_{n\geq 1} \lim_{x\to\infty} k_n(x) > 0$ and $C = \inf_{n\geq 1} \lim_{x\to 0^+} h_n(x) > 0$.

Assume $\sum_{n=1}^{\infty} E(g_n(|X_n|)) < \infty$. Then $[S(n) - E(S(n))]/f(n) \to 0$ almost surely as $n \to \infty$.

Proof The proof of Theorem 6 goes through except that we have to verify condition(c).

$$\sum_{n=1}^{\infty} \frac{1}{(f(n))^2} E((X_n^2) I(X_n \leq a_n))$$

$$\leq \frac{1}{f^2(1)} \sum_{n=1}^{\infty} E\left(\frac{X_n^2}{g_n(|X_n|)} g_n(|X_n|)\right)$$

$$\leq \frac{1}{bf^2(1)} \sum_{n=1}^{\infty} E(g_n(|X_n|)) < \infty .$$

Corollary 3 Let $\{X_n, n \geq 1\}$ be as in Theorem 8 if $\sum_{n=1}^{\infty} E(X_n^2) < \infty$, then $[S(n) - E(S(n))]/f(n) \to 0$ almost surely as $n \to \infty$.

We first provide a lemma, which can be proved using the formula of summation by parts (Chandra (1992)).

Lemma 1 If $\sum_{n=1}^{\infty} b_n < \infty$ and b_n is decreasing, then for any bounded $\{a_n, n \geq 1\}$ such that $\{na_n, n \geq 1\}$ is increasing, $\sum_{n=1}^{\infty} [na_n - (n-1)a_{n-1}] b_n < \infty$.

Theorem 9 Let $\{X_n, n \geq 1\}$ be pairwise *ND* random variables. Assume that there is a positive even and continuous function Φ on $\Re^1$ such that

(i) $t^{-1}\Phi(t)$ is increasing to ∞ as $t \to \infty$;

(ii) $\sup_{n\geq 1}[n^{-1} \sum_{i=1}^{n} E(\Phi(|X_i|))] = c(say) < \infty$;

(iii) $\sum_{n=1}^{\infty} (\Phi(n))^{-1} < \infty$.

Then $n^{-1} \sum_{i=1}^{n} (X_i - E(X_i)) \to 0$ almost surely as $n \to \infty$.

Proof We use Theorem 4 with $B_n = (-\infty, n]$ for $n \geq 1$. Put $a_n = n^{-1} \sum_{n=1}^{\infty} E(\Phi(|X_i|))$ for $n \geq 1$. We first verify condition (a);

$$\sum_{n=1}^{\infty} P(X_n > n) \leq \sum_{n=1}^{\infty} P(\Phi(|X_n| \geq \Phi(n))$$

$$\leq \sum_{n=1}^{\infty} E(\Phi|X_n|)/\Phi(n) = \sum_{n=1}^{\infty} [na_n - (n-1)a_{n-1}]/\Phi(n) < \infty$$

by lemma 1 and (iii). To prove condition (b), let $e > 0$. There is an integer $N_1 > 1$ such that for each $n \geq 1$,

$$n^{-1} \sum_{i=1}^{n} E(|X_i| I(|X_i| > N_1)) < e/2;$$

this is possible since $\{X_n, n \geq 1\}$ is Cesaro uniformly integrable (See remark 5 of Chandra (1992)). Next There is an integer $N > N_1$ such that for each $n \geq N$ $n^{-1} \sum_{i=1}^{N_1} E(|X_i|) < e/2$. Then for $n \geq N$,

$$\sum_{i=1}^{n} E(X_i I(X_i > i)) \leq \sum_{i=1}^{n} E(|X_i| I(|X_i| > i))$$

$$\leq \sum_{i=1}^{N_1} E(|X_i|) + \sum_{i=1}^{n} E(|X_i| I(|X_i| > N_1)) < ne .$$

It is clear that $B_n = C_n \cup D_n \cup E_n$ where $C_n = (-\infty, -n)$, $D_n = [-n, -n^{1/4}] \cup [n^{1/4}, n]$, and $E_n = (-n^{1/4}, n^{1/4})$. To prove condition (c), it suffices to show that

$$\sum_{n=1}^{\infty} n^{-2} E(X_n^2 I(C_n)) < \infty,$$

$$\sum_{n=1}^{\infty} n^{-2} E(X_n^2 I(D_n)) < \infty \quad \text{and}$$

$$\sum_{n=1}^{\infty} n^{-2} E(X_n^2 I(E_n)) < \infty . \qquad (9)$$

We first show that $\sum_{n=1}^{\infty} n^{-2} E(X_n^2 I(C_n)) < \infty$. For each $n \ge 1$, there is a z_n in the interval $(-\infty, -1)$ such that

$$\Phi(z_n)/z_n^2 \le 2 \inf\{y : y = \Phi(x)/x^2, x < -n\}.$$

Put $F_n = \{y : y = \Phi(x)/x^2, x < -n\}$ and $a_n = \inf_y F_n$. It is obvious that $F_{n+1} \subseteq F_n$ and $a_n \uparrow$. Hence it is possible to let $\{z_n, n \ge 1\}$ be increasing sequence. Then we have

$$\Phi(z_n)/z_n^2 \le 2\Phi(x)/x^2 \qquad \forall x < -n$$

and

$$x^2 \le 2z_n^2 \frac{\Phi(x)}{\Phi(z_n)} \le 2\frac{|z_n|}{\Phi(z_n)} \Phi(x)|z_1| \le 2\frac{|z_1|}{\Phi(z_1)} \Phi(x)|z_1|.$$

Hence

$$\sum_{n=1}^{\infty} n^{-2} E(X_n^2 I(X_n < -n)) \le 2\frac{|z_1|^2}{\Phi(z_1)} \sum_{n=1}^{\infty} n^{-2} E(\Phi(|X_n|)) < \infty.$$

To complete proving of condition (c), it suffices to show that

$$\sum_{n=1}^{\infty} n^{-2} E(X_n^2 I(D_n)) < \infty. \qquad (10)$$

For each $n \ge 1$, there is a z_n in the interval $[n^{1/4}, n]$ such that

$$\Phi(z_n)/z_n^2 \le 2 \inf\{y : y = \Phi(x)/x^2 : n^{1/4} \le x \le n\},$$

note that the right side of the above inequality is positive. Then for $x \in [n^{1/4}, n]$, we have

$$x^2 \le 2nz_n \frac{\Phi(x)}{\Phi(z_n)} \qquad (\text{as } z_n \le n)$$

$$\le 2n^2 \Phi(x)/t_n \quad (\text{by } z_n \ge n^{1/4} \text{ and } (i))$$

where $t_n = n^{3/4}\Phi(n^{1/4})$ for $n \ge 1$. Observe that

$$\sum_{n=1}^{\infty} n^{-2} E(X_n^2 I(D_n)) \le 2\sum_{n=1}^{\infty} E(\Phi(|X_n|))/t_n = \sum_{n=1}^{\infty} [na_n - (n-1)a_{n-1}]/t_n.$$

So (10) will follow if we show that $\sum_{n=1}^{\infty} 1/t_n < \infty$ (using lemma 1). For this purpose, we use Lemma 15 of Petrov (1975, 277-278) with $a_n = n^{1/4} - (n-1)^{1/4}$ for $n \ge 1$, $y(x) = \Phi(|X|)/|X|$; here we are following the notation of Petrov (1975) and using assumptions (i) and (iii). As $a_n \ge 1/(4n^{3/4})$ for each n and $t_n = ny(n^{1/4})$, we get $\sum_{n=1}^{\infty} 1/t_n < \infty$.

We finally prove condition (d). There is a $t_0 > 0$ such that $\Phi(t) \ge 1$ for each $t \ge t_0$, and so $|x| \le t_0 + \Phi(|x|)$ which implies that for each $n \ge 1$, $n^{-1}\sum_{i=1}^{n} E(|X_i|) \le t_0 + c$.

References:

[1] Amini, M., *Ph.D. Thesis,* Mashhad University, (2000).

[2] Bozorgnia, A. Patterson, R.F. and Taylor, R.L., Limit Theorems for dependent random variables, *World Congress Nonlinear Analysts*, **92,** (1996), PP.1639-1650.

[3] Chandra, T.K. and Goswami, A., Cesaro uniform integrability and the strong law of large numbers, *Sankhya*: *The Indian Journal of Statistics*, Vol. **54**, Series A, Pt.2, (1992), PP. 215-231.

[4] Csorgo, S., Tandori, K. and Totik, V., On the strong law of large numbers for pairwise independent random variables, *Acta Math. Hungarica*, **42**, (1981), PP.319-330.

[5] Chung, K.L., Note on some strong laws of large numbers, *Amer. J. Math*., **69**, (1947) 189-192 **.**

[6] Chung, K.L., *A course in probability theory,* second edition, Academic Press, New York, (1974)

[7] Landers, D. and Rogge, L., Laws of large numbers for pairwise independent uniformly integrable random variables, *Tech Report* No **98** University of Cologne, West Germany, (1986)

[8] Petrov,V.V., *Sums of independent Random Variables*, Springer-Verlag, New York-Heidelberg-Berlin, (1975)